Golden Ratio Based Fine Structure Constant and Rydberg Constant for Hydrogen Spectra

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2(Dedicated to Niels Bohr on the occasion of the centenary of his paper on hydrogen)

Abstract: The surprising discovery that Bohr radius is divided at the Golden point into two sections pertaining to the electron and proton due to electrostatic reasons led to the general finding that it is a geometrical constant in atomic and ionic radii, bond lengths and bond angles. In an earlier article, the fine structure constant was also interpreted in terms of the Golden ratio. Here experimental data on wavelengths of hydrogen spectral lines have been used to evaluate the Rydberg constant, fine structure constant and Sommerfeld’s relativity factor and are all shown to be simple functions of the Golden ratio.

Keywords: Hydrogen spectra, Fine structure constant, Golden ratio, Relativity factor, Rydberg constant, Rydberg formula, Bohr radius

1. Introduction
1.1 Golden ratio as a geometrical constant in the dimensions of atoms

Some years ago, the author made a new suggestion1 that the ground state ionization potential (Ip) of hydrogen (H), could be considered as the difference between the potentials, (Ie) and (Ia) of the oppositely charged proton (p+) and electron (e-), necessary to pull them apart against their electrostatic interaction. This is expressed by the equation,

(1)

where e is the electric charge, \( \kappa \) is the electric constant and \( a_B \) is the Bohr radius (= distance between the proton and electron) which is the sum \( a_{B,p} + a_{B,e} \), of the two sections pertaining to the proton (p+) and electron (e-). On using the data from2, \( e/2\kappa = 7.19982 \text{ Å eV} \) and \( I_H = 13.5984 \text{ eV} \), the value of \( a_B = e/2I_H = 0.052946 \text{ nm} \).

From equation (1), one obtains the following successive relations,

(2)

(3)

(4)

where \( \phi \) is called the Golden ratio or The Divine ratio3. It was thus a surprise to find1 for the first time that the Bohr radius is divided into two unique sections at the point of electrical neutrality, which is the Golden point. The Golden ratio5, which manifests itself in many spontaneous creations of Nature was thus found to originate right in the core of atoms14.

It was thus found1 that the covalent bond lengths between atoms of the same kind are also divided at the Golden point into two sections which form the cationic and anionionic radii. The additivity of atomic and ionic radii in the bond lengths in many inorganic, organic and biological molecules and in aqueous media were thus established4,5. Recently, atomic radii of all elements obtained from the lattice parameters and bond lengths were shown9,10 to be related to the respective Bohr radii obtained from first ionization potentials through simple functions of the Golden ratio. Bond angles in simple molecules like water, carbon dioxide, sulfur dioxide, etc. were also interpreted11 in terms of the Golden ratio.

1.2. Golden ratio based fine structure constant as a mathematical constant

The fine structure constant12 (\( \alpha \)), which has the value2, \( 1/137.036 = 0.0072974 \), is the ratio of the Compton wavelength \( (\lambda_{c,H}) \) to the de Broglie wavelength \( (\lambda_{db} = 2\pi a_B) \), and is also equal to the ratio of twice the de Broglie wavelength to the Rydberg wavelength \( ( = \lambda_H = R_H^{-1}) \), where \( R_H \) is the Rydberg constant.)
\[
\alpha = \lambda_{C,H}/\lambda_{AB} = 2\lambda_{AB}/\lambda_H = 0.0072974 = 2.627/360 \tag{6}
\]

As \(\lambda_{AB} = 2\pi a_B = 0.3327 \text{ nm}\) is the circumference of the Bohr circle, the Compton wavelength is the length of a small arc, \(\lambda_{C,H} = \alpha\lambda_{AB} = 0.00243 \text{ nm}\) on the Bohr circle, which subtends an angle 2.627° at the center of the Bohr circle. Similarly, \(2\lambda_{AB} = 0.6653 \text{ nm}\) is equivalent to the length of an arc of a circle subtending an angle 2.627° at the center of a circle of circumference, \(\lambda_H = 91.1756 \text{ nm}\) and radius = \(\lambda_H/2\pi = 14.511 \text{ nm}\). On noticing \(^{12}\) the closeness of the fine structure constant, \(\alpha = 2.627/360\) to the ratio of the angles, \(\phi/360 = 2.618/360\) the author suggested that the small difference, 2.627° - 2.618° = 0.009° could be due to the Sommerfeld’s relativity correction factor\(^ {12}\),

\[
\alpha - (1- \gamma)/\gamma = (\phi/360) = \alpha_o \tag{7}
\]

where \(\gamma = (1 - \alpha^2)^{1/2}\). For \(\alpha = 0.0072974\), \((1- \gamma)/\gamma \approx 0.0096/360 = 0.0000267\) and \(\alpha - \alpha_0 = 0.0072974 - 0.0072273 = 0.0000251\).

It was also pointed out \(^{12}\) that the ratio \(360/\phi^2 = 137.508\), which is a Golden section of 360, differs from the inverse fine structure constant \(^{2}\) \((\alpha^{-1} = 137.036)\) by \(2/\phi^2 = 0.472\) probably due to the difference in the g-factors for the electron and proton, \(g_e\) and \(g_p\) arising from the magnetic momenta of the two particles. Thus,

\[
360/\phi^2 - \alpha^{-1} = (g_p - g_e)/(g_p + g_e) = 2(a_{B,e} - a_{B,p})/a_B = 2/\phi^2 = 0.472 \tag{8a}
\]

\[
\alpha^{-1} = 360 - 2\phi^2/\phi^2 = 137.036 \tag{8b}
\]

\[
\alpha = \phi^2/(360 - 2\phi) = 0.0072974 \tag{8c}
\]

As per equation (8b), where \(360 - 2/\phi = 358.764\), the Golden ratio based value of \(\alpha^{-1} = 137.0356\) rounds up to the value, 137.036 in\(^{12}\). All details and a descriptive figure can be found in\(^{12}\). Thus, equations (8b) and (8c) represent the fine structure constant as a mathematical constant in terms of the Golden ratio.

2. Present work
2.1. The Rydberg constant evaluated from measured wavelengths of hydrogen spectrum.

In this work, the measured wavelengths\(^ {14}\) of the hydrogen spectrum (see column 2, Table 1) have been used to evaluate the inverse Rydberg constant, \(R_H^{-1} = \lambda_H\) (which can be called the Rydberg wavelength) using the Rydberg equation,

\[
1/\lambda = R_H[(1/n_1^2) - (1/n_2^2)] \tag{9}
\]

\[
R_H = 1/\lambda_H = \alpha/4\pi a_B \tag{10}
\]

where \(n_1\) and \(n_2\) (\(> n_1\)) are the principal quantum numbers and \(a_B = e/2\kappa H = 0.052946 \text{ nm}\) from equation (1). On rewriting Rydberg equation (9) in the form,

\[
\lambda[(1/n_1^2) - (1/n_2^2)] = \lambda_H = (4\pi a_B/\alpha) \tag{11}
\]

one can see that the product on the left side must be a constant equal to the Rydberg wavelength, \(\lambda_H\), for all wavelengths, \(\lambda\). These products corresponding to all the wavelengths (in column 2) are given in column 6 in Table 1. For \(a_B = 0.052946 \text{ nm}\) and \(\alpha^{-1} = 137.036\), \(\lambda_H = (4\pi a_B/\alpha) = 91.175 \text{ nm}\). It can be seen from the products in column 6 that this value of \(\lambda_H\) is nearly reproduced by the Lyman series, but the values for Balmer and Paschen series are slightly lower. For the wavelengths for the Bracket and Pfund series, \(\lambda_H\) values are not the same. A graph of \(1/\lambda\) vs \([(1/n_1^2) - (1/n_2^2)]\) (excluding the last two series) in Figure 1 here gives the slope, \(R_H = 1.09670 \times 10^7 \text{ m}^{-1}\), and its reciprocal, \(\lambda_H = 91.186 \text{ nm}\). This least squares value is given in the top of column 6.

2.2. The fine structure constant, relativity factor and Golden ratio derived from the Rydberg constant

The values of the fine structure constant, \(\alpha\) obtained as the ratio, \(4\pi a_B/\lambda_H\), as per equation (11), using the \(\lambda_H\) values in column 6, are given in column 7. But for the Lyman series, these values also differ slightly from the value, 1/137.036 = 0.0072974.

In column 8 are given the values of \((1- \gamma)/\gamma = \alpha_{rel}\), where \(\gamma = (1 - \alpha^2)^{1/2}\) were calculated from the \(\alpha\) values in column 7, as per equation (7). The difference, \(\alpha - \alpha_{rel} = \alpha_0\) is given in column 9. Column 10 gives the calculated values of \(360a_o = \phi^2\). It can be seen that these values of \(\phi^2\), evaluated from the Rydberg wavelength in column 6, are close to 2.618, in accordance with equation (7).

2.3. The Rydberg equation in terms of the Golden ratio

On using the Golden ratio based equation (8b) for \(\alpha = \phi^2/(360 - 2\phi),\) the Rydberg equation (9) can be written in terms of the Golden ratio as,

\[
1/\lambda = [\phi^2/(360 - 2\phi)(4\pi a_B)][(1/n_1^2) - (1/n_2^2)] \tag{12a}
\]
R_B = (a/(4πa_B)) = [φ^2/(360 - 2φ)](4πa_B) 
(12b)

where a_B/φ^2 = a_B = 0.02022 nm (see equation 5b) is the proton Bohr radius pertaining to the proton and 2πa_B = λ_{AB,p} = a_B/φ^2 is the circumference of the circle with radius a_B, with the proton at the center. λ_{AB}/φ^2 is also the Golden arc of the Bohr circle, subtending an angle, 360/φ^2 at the center.

Summary: The Golden ratio based fine structure constant and Rydberg formula are given by equations (8b) and (12a,b) respectively.

References
[14] http://hyperphysics.phy-astr.gsu.edu/Hbase/tables/hydspec.html#61

Figure Legend.
Figure 1. Linear dependence of (1/λ) on [(1/n_1^2) - (1/n_2^2)] for the experimental data in for the Lyman, Balmer and Paschen series.
Table 1. Measured wavelengths (λ) of hydrogen spectra\textsuperscript{14}, Rydberg wavelength (λ\textsubscript{R}), de Broglie wavelength (λ\textsubscript{dB} = 2πa\textsubscript{B}), fine structure constant (α), relativity factor [(1-γ)/γ] and square of the Golden ratio (ϕ\textsuperscript{2}).

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