Clique Complex Homology: A Combinatorial Invariant for Chordal Graphs

Allen D. Parks

Abstract: It is shown that a geometric realization of the clique complex of a connected chordal graph is homologically trivial and as a consequence of this it is always the case for any connected chordal graph $G$ that $\sum_{k=1}^{\omega(G)}(-1)^{k-1}\eta_k(G) = 1$, where $\eta_k(G)$ is the number of cliques of order $k$ in $G$ and $\omega(G)$ is the clique number of $G$.

Keywords: algebraic graph theory; chordal graph; clique complex; hypergraph; homology; Mayer-Vietoris theorem; graph invariant; Euler-Poincaré formula

1. Introduction
In recent years much attention has been devoted to the study of chordal graphs. In addition to understanding their intrinsic properties from a graph theoretic perspective, a major motivation for this research has been their usefulness in such diverse applied domains as biology, relational database design, computer science, matrix manipulation, psychology, scheduling, and genetics, e.g., [1, 2]. Graph invariants have also been the subject of intense research because of their utility for determining class membership of graphs and recognizing non-isomorphic graphs, e.g., [3].

In this short paper a new combinatorial invariant for connected chordal graphs is found using the well-established relationship between a connected chordal graph $G$ and its associated $\alpha$-acyclic hypergraph $\mathcal{H}(G)$. The fact that $\mathcal{H}(G)$ is $\alpha$-acyclic and that $\mathcal{H}(G)$ also generates the clique complex $\mathcal{C}(G)$ of $G$ allows a straightforward application of the Mayer-Vietoris Theorem to show that a geometric realization $|\mathcal{H}|$ of $\mathcal{C}(G)$ is homologically trivial. The combinatorial invariant for connected chordal graphs follows from the direct application of the Euler-Poincare formula to $|\mathcal{H}|$.

The remainder of this paper is organized as follows: The relevant definitions and terminology are summarized in the next section. Required preliminary lemmas are provided in Section 3 and the main results are established in Section 4. A simple illustrative example is presented in Section 5. Closing remarks comprise the final section of this paper.

2. Definitions and Terminology
A graph $G$ is a pair $(V(G), E(G))$, where $V(G)$ is a finite non-empty set of vertices and $E(G)$ is either a set of doubleton subsets of $E(G)$ called edges or the empty set $\emptyset$. The order of $G$ is the cardinality of $V(G)$ and the size of $G$ is the cardinality of $E(G)$. Two vertices $u, v \in V(G)$ are adjacent when $e = \{u, v\} \in E(G)$ in which case $e$ is said to join $u$ and $v$. A $u$-$v$ walk is an alternating sequence of vertices and edges beginning with $u$ and ending with $v$ such that every edge joins the vertices immediately preceding and following it. A $u$-$v$ path is a $u$-$v$ walk in which no vertex is repeated. In this case $u$ is said to be connected to $v$. $G$ is connected if its order is one or if every two vertices in $G$ are connected. A $u$-$v$ path for which $u = v$ and which contains at least three edges is a cycle. The length of a cycle is the number of edges contained within it and a chord of a cycle is an edge between nonconsecutive vertices in the cycle.

A graph is a chordal graph if every cycle of length at least four has a chord. A graph is complete if every two of its vertices are adjacent. A graph $F$ is a subgraph of $G$ if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A clique in $G$ is either a vertex or a complete subgraph of $G$ and is maximal if it is not a proper subgraph of another clique. The order of a clique is the cardinality of its vertex set and the clique number $\omega(G)$ of $G$ is the maximum order among the maximal cliques of $G$. A hypergraph $\mathcal{H}$ is a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite set of vertices and $\mathcal{E}$ is a set of hyperedges which are non-empty subsets of $\mathcal{N}$ (it is hereafter assumed that $\mathcal{H}$ is reduced, i.e., no hyperedge of $\mathcal{H}$ is a subset of another hyperedge of $\mathcal{H}$). The hypergraph $\mathcal{H}(G)$ associated with graph $G$ has $V(G)$ as its vertices and the hyperedges of $\mathcal{H}(G)$ are the sets of vertices in the maximal cliques of $G$. $\mathcal{H}(G)$ is connected if $G$ is connected. A connected hypergraph is $\alpha$-acyclic if it has the (nonempty) running intersection property, i.e., if there is an ordering $(e_1, e_2, \ldots, e_n)$ of its hyperedges so that for each $i, 2 \leq i \leq n$, there is a $j_i < i$ such that...
\[ \emptyset \neq (\varepsilon_i \cap (U_{k \leq i} \varepsilon_k)) \subseteq \varepsilon_{j_i} \text{ (note the fact that each of these } n - 1 \text{ intersections is non-empty implies connectedness) [4, 5].} \]

The **closure** \( \text{Cl}(S) \) of a finite set \( S \) is the family of nonempty subsets of \( S \). The closure \( \text{Cl}(H) \) of a hypergraph \( H \) is the union of the closure of each of its hyperedges, i.e., \( \text{Cl}(H) = \bigcup_{e \in e} \text{Cl}(e) \), and the closure \( \text{Cl}(G) \) of a graph \( G \) is \( \text{Cl}(G) = \bigcup_{V \subseteq V} \text{Cl}(V) \), where \( V \) is the set of vertices in a maximal clique of \( G \) and \( M \) is the set of all such \( V \)'s.

Let \( \{a_0, a_1, \ldots, a_k\} \) be a set of geometrically independent points in \( \mathbb{R}^m \). The **\( k \)-simplex** (or **simplex**) \( \sigma^k \) spanned by \( \{a_0, a_1, \ldots, a_k\} \) is the set of points \( x \in \mathbb{R}^m \) for which there exist non-negative real numbers \( \lambda_0, \lambda_1, \ldots, \lambda_k \) such that \( x = \sum_{i=0}^k \lambda_i a_i \) and \( \sum_{i=0}^k \lambda_i = 1 \). In this case \( \{a_0, a_1, \ldots, a_k\} \) is the vertex set of \( \sigma^k \). A face of \( \sigma^k \) is any simplex spanned by a non-empty subset of \( \{a_0, a_1, \ldots, a_k\} \). A **finite geometric simplicial complex** (or complex) \( K \) is a finite union of simplices such that: (i) every face of a simplex of \( K \) is in \( K \); and (ii) the non-empty intersection of any two simplices of \( K \) is a common face of each. The **dimension** \( \text{dim}(K) \) of \( K \) is the largest positive integer \( m \) such that \( K \) contains an \( m \)-simplex. The **vertex scheme** of \( K \) is the family of all vertex sets which span the simplices of \( K \). If \( \{l_i : i \in I\} \) is a collection of subcomplexes of \( K \), then \( \bigcup_{i \in I} l_i \) and \( \cap_{i \in I} l_i \neq \emptyset \) are also subcomplexes of \( K \).

A **finite abstract simplicial complex** (or **abstract complex**) is a finite collection \( S \) of finite non-empty sets such that if \( A \) is in \( S \), then so is every non-empty subset of \( A \). Thus, the vertex scheme of a complex is an abstract complex - as are finite unions of set closures and finite intersections of set closures. The abstract complex \( \text{Cl}(G) \) of graph \( G \) is called the **clique complex** \( \mathcal{C}(G) \) of graph \( G \).

Two abstract complexes \( S \) and \( T \) are **isomorphic** if there is a bijection \( \varphi \) from the vertex set of \( S \) onto the vertex set of \( T \) such that \( \{a_0, a_1, \ldots, a_k\} \in S \) if, and only if, \( \{\varphi(a_0), \varphi(a_1), \ldots, \varphi(a_k)\} \in T \). Every abstract complex \( S \) is isomorphic to the vertex scheme of some geometric simplicial complex \( K \), in which case \( K \) is the geometric **realization** of \( S \) and is uniquely determined (up to linear isomorphism). An isomorphism between \( S \) and the vertex scheme of \( K \) is denoted \( S \cong K \). An **edge** in \( S \) is a doubleton subset of vertices contained in \( S \). A distinct pair of vertices \( u, v \) of \( S \) are **path connected** if there is an alternating sequence \( u(u, x_1)x_1(x_2, x_3) \cdots (x_p, v) \) of vertices and edges of \( S \). The abstract complex \( S \) is **connected** when either \( S \) has one vertex or all its pairs of vertices are path connected. If \( S \) is connected, then so is any geometric realization of \( S \).

To each (simplicial) complex \( K \) there corresponds a **chain complex**, i.e., abelian groups \( \Gamma_p(K) \) and homomorphisms \( \partial_{p+1}: \Gamma_{p+1}(K) \to \Gamma_p(K), p \geq 0 \). If \( K \) is finite and \( \xi_p(K) \) is the number of \( p \)-simplices in \( K \), then the **rank of** \( \Gamma_p(K) \) is \( \xi_p(K) \), i.e., \( \Gamma_p(K) \) is isomorphic to (denoted \( " \cong " \)) the direct sum \( " \bigoplus " \) of \( \xi_p(K) \) copies of the additive group of integers \( \mathbb{Z} \). The \( p \)-th homology group of \( \mathbb{K} \) is the quotient group \( H_p(\mathbb{K}) = \text{ker} \partial_p / \text{im} \partial_{p+1} \) and its rank is the \( p \)-th Betti number \( b_p(\mathbb{K}) \). If \( K \) is connected, then \( H_0(\mathbb{K}) \cong \mathbb{Z} \) and \( K \) is homologically trivial if it is connected and \( H_1(\mathbb{K}) \cong 0 \), \( p \geq 1 \), where \( 0 \) is the trivial group. The complex of a simplex is homologically trivial.

### 3. Preliminary Lemmas

Several lemmas are required to prove the main results found in the next section. The first three are well known and are repeated here for completeness. The fourth is lesser known and is due to D’Atri et al.

**Lemma 3.1.** [6] (Euler-Poincaré) If \( K \) is a complex of finite dimension, then

\[ \sum_{p=0}^{\text{dim}(K)} (-1)^p \xi_p(K) = \sum_{p=0}^{\text{dim}(K)} (-1)^p b_p(K). \]

**Lemma 3.2.** [7] (Mayer-Vietoris) Let \( K \) be a complex with subcomplexes \( K_0 \) and \( K_1 \) such that \( K = K_0 \cup K_1 \). Then there is an exact sequence

\[ \cdots \to H_p(K_0 \cap K_1) \to H_p(K_0) \oplus H_p(K_1) \to H_p(K) \to H_{p-1}(K_0 \cap K_1) \to \cdots \]

**Lemma 3.3.** [8] Let \( A \) be an abelian group, \( F \) be a free abelian group, and \( \theta: A \to F \) be a surmorphism. Then \( A \cong \text{ker} \theta \oplus F \).

**Lemma 3.4.** [9] If \( G \) is a connected chordal graph, then \( \mathcal{H}(G) \) is a-acyclic.

The closure operation \( \text{Cl} \) is also important for proving the main results. The following three lemmas provide the required key properties of \( \text{Cl} \). Since the next lemma is straightforward, its proof has been omitted.

**Lemma 3.5.** Let \( \{S_i : i \in I\} \) be a collection of non-empty finite sets. Then the following statements are true:

1. \( \bigcup_{i \in I} S_i \subseteq \text{Cl}(A) \) if, and only if, \( \text{Cl}(A) \subseteq \text{Cl}(B) \).
2. \( \bigcap_{i \in I} \text{Cl}(S_i) = \text{Cl}(\bigcap_{i \in I} S_i) \).
3. \( \bigcup_{i \in I} \text{Cl}(S_i) = \text{Cl}(\bigcup_{i \in I} S_i) \).
4. \( A \cap B \neq \emptyset \) if, and only if, \( \text{Cl}(A) \cap \text{Cl}(B) \neq \emptyset \); and
5. \( A \neq \emptyset \) if, and only if, \( \text{Cl}(A) \neq \emptyset \).

**Lemma 3.6.** If the ordering of sets \( (S_1, S_2, \ldots, S_n) \) has the running intersection property, then the ordering \( \{\text{Cl}(S_1), \text{Cl}(S_2), \ldots, \text{Cl}(S_n)\} \) also has the running intersection property.

**Proof.** The proof results from use of the identity \( S_i \cap (\bigcup_{k \leq l-i} S_k) = \bigcup_{k \leq l-i} (S_i \cap S_k) \) in the following
implication chain: $\emptyset \neq S_i \cap (U_{k<i} S_k) \subseteq S_j \implies \emptyset \neq U_{k<i} (S_i \cap S_k) \subseteq S_j \implies \emptyset \neq CL(U_{k<i} (S_i \cap S_k)) \subseteq CL(S_j) \quad$ (Lemma 3.5 (1), (5)) $\implies \emptyset \neq U_{k<i} (CL(S_i) \cap CL(S_k)) \subseteq CL(S_j) \quad$ (Lemma 3.5 (3)) $\implies \emptyset \neq U_{k<i} (CL(S_i) \cap CL(S_k)) \subseteq CL(S_j) \quad$ (Lemma 3.5 (2)) $\implies \emptyset \neq CL(S_i) \cap (U_{k<i} CL(S_k)) \subseteq CL(S_j)$. Thus, the $CL$ operation preserves the running intersection property.

**Lemma 3.7.** Let the ordering of sets $(S_1, S_2, \ldots, S_n)$ have the running intersection property. If $S' = S_i \cap (U_{k<i} S_k)$, then $CL(S') = CL(S_i) \cap (U_{k<i} CL(S_k))$.

**Proof.** Again use the identity $S' = S_i \cap (U_{k<i} S_k)$ = $U_{k<i} (S_i \cap S_k)$ and let $A \neq \emptyset$. Application of Lemma 3.5 and the definition of $CL$ provides the following biconditional chain and completes the proof: $A \in CL(S') \iff A \subseteq S' \iff A \subseteq S_i \cap S_k$ for some $k < i \iff A \in CL(S_i) \cap CL(S_k)$ for some $k < i \iff A \in U_{k<i} CL(S_i) \cap (U_{k<i} CL(S_k)) \iff A \in CL(S_i) \cap (U_{k<i} CL(S_k))$. Thus, $CL(S_i) = CL(S_i) \cap (U_{k<i} CL(S_k))$.

**4. Main Results**

**Lemma 4.1.** $CL(\mathcal{H}(G))$ and $C(G)$ are identical abstract complexes.

**Proof.** By definition each hyperedge $\varepsilon$ in $\mathcal{H}(G)$ is the vertex set $\mathcal{V}$ of a maximal clique in $G$ so that $CL(\varepsilon) = CL(\mathcal{V})$ (Lemma 3.5 (1)). Because of this one-to-one correspondence between hyperedges and maximal cliques, it must therefore be the case that $CL(\mathcal{H}(G)) = \bigcup_{\varepsilon \in \mathcal{E}} CL(\varepsilon) = \bigcup_{\varepsilon \in \mathcal{E}} CL(\mathcal{V}) = CL(\mathcal{V}) \equiv C(G)$.

**Corollary 4.2.** $CL(\mathcal{H}(G))$ and $C(G)$ have the same geometric realizations.

**Proof.** Let $\mathbb{K}$ be the geometric realization of $CL(\mathcal{H}(G))$ in which case $CL(\mathcal{H}(G)) \equiv \mathbb{K}$. Since $C(G) = CL(\mathcal{H}(G))$ (Lemma 4.1), it must also be the case that $C(G) \equiv \mathbb{K}$.

**Lemma 4.3.** Let $G$ be a connected chordal graph, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be an ordering of the hyperedges of $\mathcal{H}(G)$ which exhibit the running intersection property, and $\mathbb{K}$ be a geometric realization of $C(G)$. Then:

1. there is a subcomplex $\mathbb{K}_i$ of $\mathbb{K}$ such that $CL(\varepsilon_i) \equiv \mathbb{K}_i$;
2. the subcomplex $\mathbb{L}_i$ of $\mathbb{K}$ such that $U_{k<i} CL(\varepsilon_k) \equiv \mathbb{L}_i$ is connected;
3. $\mathbb{K}_i \cap \mathbb{L}_i$ is the complex of a simplex of $\mathbb{K}$; and
4. $\mathbb{K}_i \cup \mathbb{L}_i$ is a connected subcomplex of $\mathbb{K}$.

**Proof.** Item (1) follows from Corollary 4.2; the one-to-one correspondence between maximal cliques in $G$ and hyperedges in $\mathcal{H}(G)$; the fact that for each hyperedge $\varepsilon_i$ there is a set $\mathcal{V}$ of maximal clique vertices spanning a simplex in $\mathbb{K}$ such that $\varepsilon_i = \mathcal{V}$ $\Rightarrow$ $CL(\varepsilon_i) = CL(\mathcal{V})$ (Lemma 3.5 (1)); and that the abstract complexes $CL(\varepsilon_i) = CL(\mathcal{V})$ are therefore both isomorphic to the vertex scheme of the same simplex. Item (2) follows from an induction argument on $i \geq 2$ and the running intersection property’s nonempty condition $\emptyset \neq CL(\varepsilon_i) \cap (U_{k<i} CL(\varepsilon_k))$, along with the one-to-one correspondence between hyperedges and vertex sets of maximal cliques. Item (3): Since $\mathbb{K}_i$ and $\mathbb{L}_i$ are subcomplexes of $\mathbb{K}$, then so is $\mathbb{K}_i \cap \mathbb{L}_i$. Because $\mathcal{H}(G)$ has the running intersection property, along with Lemma 4.1, Corollary 4.2, and Lemma 3.7, it follows that $\emptyset \neq CL(\varepsilon_i) \cap (U_{k<i} CL(\varepsilon_k)) = \mathbb{K}_i \cap \mathbb{L}_i$. Also, since $\mathbb{K}_i$ and $\mathbb{L}_i$ are subcomplexes of $\mathbb{K}$, then so is $\mathbb{K}_i \cup \mathbb{L}_i$. Thus, $\mathbb{K}_i \cup \mathbb{L}_i$ is a connected simplex of $\mathbb{K}$ (item (1)) and $\mathbb{K}_i \cap \mathbb{L}_i$ has the complex of a simplex in common (item (3)), then $\mathbb{K}_i \cup \mathbb{L}_i$ must be connected.

**Theorem 4.4.** If $G$ is a connected chordal graph and $\mathbb{K}$ is a geometric realization of $C(G)$, then $\mathbb{K}$ is homologically trivial.

**Proof.** Since $\mathbb{K}$ is a geometric realization of $C(G)$, it is also a geometric realization of $CL(\mathcal{H}(G))$ (Corollary 4.2) and since $G$ is a connected chordal graph, then $\mathcal{H}(G)$ is an $\alpha$ -acyclic hypergraph (Lemma 3.4). Consequently, there is an ordering $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of the hyperedges of $\mathcal{H}(G)$ which exhibit the running intersection property. Let $\mathbb{K}_i$ and $\mathbb{L}_i, i \in \{2,3,\ldots,n\}$, be subcomplexes of $\mathbb{K}$ such that $CL(\varepsilon_1) = \mathbb{K}_i$ and $U_{k<i} CL(\varepsilon_k) = \mathbb{L}_i$.

in which case $\mathbb{K}_i \cap \mathbb{L}_i$ is the complex of a simplex (Lemma 4.3 (3)) and $\mathbb{K}_i \cup \mathbb{L}_i \equiv \mathbb{K}_{i+1}$ is a connected complex (Lemma 4.3 (4)). It follows that there is an associated exact sequence (Lemma 3.2) $\cdots \to H_p(\mathbb{K}_i \cap \mathbb{L}_i) \to H_p(\mathbb{K}_i) \oplus H_p(\mathbb{L}_i) \to H_p(\mathbb{K}_{i+1}) \to H_{p-1}(\mathbb{K}_i \cap \mathbb{L}_i) \to \cdots$.

Since $\mathbb{K}_i$ and $\mathbb{K}_i \cap \mathbb{L}_i$ are complexes of simplices (Lemma 4.3 (1) and (3)), then for $p \geq 1$ $H_p(\mathbb{K}_i) \equiv H_p(\mathbb{K}_i \cap \mathbb{L}_i) \equiv 0$ and for $p = 0$ $H_0(\mathbb{K}_i) \equiv H_0(\mathbb{K}_i \cap \mathbb{L}_i) \equiv \mathbb{Z}$. Substitution of these group isomorphisms into the above exact sequence induces the following two
relevant exact sequences:

0 \to H_p(\mathbb{L}_1) \to H_p(\mathbb{L}_{i+1}) \to 0 \quad (p \geq 2)

and

\cdots \to 0 \to H_1(\mathbb{L}_1) \to H_1(\mathbb{L}_{i+1}) \to Z \to Z \oplus H_0(\mathbb{L}_1) \\
\theta \to H_0(\mathbb{L}_{i+1}) \to 0 \to \cdots .

Since \( i \geq 2 \) is arbitrary, the exactness of the first induced sequence provides the isomorphism chain

\[ H_1(\mathbb{L}_2) \cong H_1(\mathbb{L}_3) \cong \cdots \cong H_p(\mathbb{L}_{n+1}) = H_p(\mathbb{K}) \quad (p \geq 2). \]

However, for \( p \geq 2 \), \( H_p(\mathbb{L}_2) \equiv 0 \) (because \( \mathbb{L}_2 \) is the complex of a simplex) and it is concluded from the isomorphism chain that

\[ H_p(\mathbb{K}) \equiv 0 \quad (p \geq 2). \]

The exactness of the second induced sequence insures that \( \theta \) is a surmorphism. Also, since \( \mathbb{L}_{i+1} \) is connected (Lemma 4.3 (4)), then \( H_0(\mathbb{L}_{i+1}) \equiv \mathbb{Z} \) and it is therefore a free abelian group. It follows from this, the fact that \( \mathbb{Z} \oplus H_0(\mathbb{L}_i) \) is abelian, and Lemma 3.3 that

\[ \mathbb{Z} \oplus H_0(\mathbb{L}_i) \equiv \ker \theta \oplus H_0(\mathbb{L}_{i+1}) \equiv \ker \theta \oplus \mathbb{Z} . \]

Using the fact that \( H_0(\mathbb{L}_i) \equiv \mathbb{Z} \) (because \( \mathbb{L}_i \) is a connected complex) and \( \ker \theta = \im \pi \) (from the exactness of the sequence) in the last expression yields

\[ \mathbb{Z} \oplus \mathbb{Z} \equiv \im \pi \oplus \mathbb{Z} \]

which implies that \( \im \pi \equiv \mathbb{Z} \). Thus, \( \pi \) is a monomorphism so that \( \ker \pi = 0 = \im \delta \) and exactness at \( \mathbb{Z} \) in the second sequence yields the exact sequence

\[ 0 \to H_1(\mathbb{L}_1) \to H_1(\mathbb{L}_{i+1}) \to 0 \]

which reveals that

\[ H_1(\mathbb{L}_1) \cong H_1(\mathbb{L}_{i+1}). \]

Since \( i \geq 2 \) is arbitrary, the following isomorphism chain holds:

\[ H_1(\mathbb{L}_2) \cong H_1(\mathbb{L}_3) \cong \cdots \cong H_1(\mathbb{L}_{n+1}) = H_1(\mathbb{K}) . \]

It can be concluded from this that since \( \mathbb{L}_2 \) is the complex of a simplex (because \( \text{CL}(\varepsilon_1) \equiv \mathbb{L}_2 \) and Lemma 4.3 (1)), then \( H_1(\mathbb{L}_2) \equiv 0 \) and

\[ H_1(\mathbb{K}) \equiv 0. \]

Consequently, \( \mathbb{K} \) is homologically trivial since it is connected (Lemma 4.3 (4)) and \( H_p(\mathbb{K}) \equiv 0, p \geq 1. \)

**Theorem 4.5.** If \( G \) is a connected chordal graph and \( \eta_k(G) \) is the number of cliques of order \( k \) in \( G \), then

\[ \sum_{k=1}^{\omega(G)} (-1)^{k-1} \eta_k(G) = 1. \]

**Proof.** Let \( \mathbb{K} \) be a geometric realization of the clique complex of \( G \). Since \( G \) is connected and chordal, then from Theorem 4.4, \( \mathbb{K} \) is homologically trivial. This implies that \( b_0(\mathbb{K}) = 1 \) and \( b_p(\mathbb{K}) = 0, p \geq 1 \), in which case Lemma 3.1 yields

\[ \sum_{p=0}^{\dim(\mathbb{K})} (-1)^p \xi_p(\mathbb{K}) = 1. \]

The one-to-one correspondence between the \( k - 1 \) dimensional simplices of \( \mathbb{K} \) and the cliques of order \( k \) in \( G \) implies \( \xi_{k-1}(\mathbb{K}) = \eta_k(G) \) and \( \dim(\mathbb{K}) = \omega(G) - 1 \). The proof is completed by using these identities in the last expression and resuming over \( k, 1 \leq k \leq \omega(G) \).

### 5. Example

In order to illustrate the theory developed above, consider the connected chordal graph \( G \), its associated hypergraph \( \mathcal{H}(G) \), and a geometric realization of \( C(G) = \text{CL}(\mathcal{H}(G)) \) shown in Figure 1. By inspection it is seen that \( \omega(G) = 4 \); the hyperedges of \( \mathcal{H}(G) \) are

\[ \varepsilon_1 = \{ a, b, c, d \}, \varepsilon_2 = \{ b, c, e \}, \varepsilon_3 = \{ e, f \}; \quad \text{and} \quad \mathcal{H}(G) \]

is \( \alpha \)-acyclic since the hyperedge ordering \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \) exhibits the (nonempty) running intersection property, i.e.,

\[ \varepsilon_3 \cap (\cup_{k<3} \varepsilon_k) = \varepsilon_3 \cap (\varepsilon_1 \cup \varepsilon_2) = \{ e, f \} \cap \{ a, b, c, d, e \} = \{ e \} \subset \varepsilon_2 \]

and

\[ \varepsilon_2 \cap (\cup_{k<2} \varepsilon_k) = \varepsilon_2 \cap \varepsilon_1 = \{ b, c, e \} \cap \{ a, b, c, d \} = \{ b, c \} \subset \varepsilon_1 . \]

The closures of these hyperedges are

\[ \text{CL}(\varepsilon_1) = \{ \{ a, b, c, d \}, \{ a, b, c \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \}, \{ a, b \}, \{ a, c \}, \{ a, d \}, \{ b, c \}, \{ b, d \}, \{ c, d \}, \{ a \}, \{ b \}, \{ c \}, \{ d \} \} , \]

\[ \text{CL}(\varepsilon_2) = \{ \{ b, c, e \}, \{ b, e \}, \{ c, e \}, \{ b \}, \{ c \}, \{ e \} \} , \]

and

\[ \text{CL}(\varepsilon_3) = \{ \{ e, f \}, \{ e \}, \{ f \} \} \]

(these closures are easily seen to validate Lemma 3.6). Since \( C(G) = \text{CL}(\mathcal{H}(G)) = \text{CL}(\varepsilon_1) \cup \text{CL}(\varepsilon_2) \cup \text{CL}(\varepsilon_3) \), then the clique complex of \( G \) is the set

\[ \{ \{ a, b, c, d \}, \{ a, b, c \}, \{ a, b, d \}, \{ a, c, d \}, \{ b, c, d \}, \{ a, c \}, \{ a, d \}, \{ b, c \}, \{ b, d \}, \{ c, d \}, \{ a \}, \{ b \}, \{ c \}, \{ d \} \} . \]
\( \mathcal{C}(G) = \{(a, b, c, d), (a, b, d), (a, c, d), (b, c, d), (b, c, e), (a, b), (a, c), (a, d), (b, c), (b, d), (c, d), (b, e), (c, e), (e, f), (a), (b), (c), (d), (e), (f)\} \)

**Figure 1.** A chordal graph \( G \), its associated hypergraph \( \mathcal{H}(G) \), and a geometric realization of \( \mathcal{C}(G) \).

Theorem 4.4 is validated by observing that since it contains no “holes”, the geometric realization of \( \mathcal{C}(G) \) in Figure 1 is homologically trivial (or somewhat more formally, there are no linear combinations of \( k \)-simplices in \( \mathcal{C}(G) \) which form \( k \)-cycles that enclose “holes”). Theorem 4.5 is validated by inspection it is easily determined that \( \eta_1(G) = 6 \), \( \eta_2(G) = 9 \), \( \eta_3(G) = 5 \), and \( \eta_4(G) = 1 \) in which case

\[
\sum_{k=1}^{4} (-1)^{k-1} \eta_k(G) = 6 - 9 + 5 - 1 = 1 .
\]

**6. Concluding Remarks**

It has been shown that any connected chordal graph has a clique complex that is isomorphic to the vertex scheme of (up to linear isomorphism) a unique homologically trivial geometric realization. As a consequence of this, the Euler-Poincaré formula provides a combinatorial invariant based simply upon the number of cliques in the graph and whose unit value holds for all connected chordal graphs. Accordingly, the contrapositive version of Theorem 4.5 serves to determine when a graph is not a connected chordal graph.

It is also interesting to note that since a tree is a connected chordal graph, then Theorem 4.5 subsumes the well-known fact that if a graph is a tree, then the difference between its order and its size is one (more specifically, if \( G \) is a tree, then \( \omega(G) = 2 \), \( \eta_1(G) \) is its order, and \( \eta_2(G) = \eta_1(G) - 1 \) is its size so that

\[
\sum_{k=1}^{2} (-1)^{k-1} \eta_k(G) = \eta_1(G) - \eta_2(G) = \eta_1(G) - (\eta_1(G) - 1) = 1 .
\]

**Acknowledgments**

This work was supported by a grant from the Naval Surface Warfare Center Dahlgren Division’s In-house Laboratory Independent Research program.

**References**