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Clique Complex Homology: A Combinatorial Invariant for Chordal Graphs

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Abstract: It is shown that a geometric realization of the clique complex of a connected chordal graph is homologically trivial and as a consequence of this it is always the case for any connected chordal graph G that $\sum_{k=1}^{\omega(G)}(-1)^{k-1}\eta_k(G)=1$, where $\eta_k(G)$ is the number of cliques of order k in G and $\omega(G)$ is the clique number of G.

Keywords: algebraic graph theory; chordal graph; clique complex; hypergraph; homology; Mayer-Vietoris theorem; graph invariant; Euler-Poincaré formula

1. Introduction

In recent years much attention has been devoted to the study of chordal graphs. In addition to understanding their intrinsic properties from a graph theoretic perspective, a major motivation for this research has been their usefulness in such diverse applied domains as biology, relational database design, computer science, matrix manipulation, psychology, scheduling, and genetics, e.g., [1, 2]. Graph invariants have also been the subject of intense research because of their utility for determining class membership of graphs and recognizing non-isomorphic graphs, e.g., [3].

In this short paper a new combinatorial invariant for connected chordal graphs is found using the well-established relationship between a connected chordal graph G and its associated α-acyclic hypergraph $\mathcal{H}(G)$. The fact that $\mathcal{H}(G)$ is α-acyclic and that $\mathcal{H}(G)$ also generates the clique complex $\mathcal{C}(G)$ of G allows a straightforward application of the Mayer-Vietoris Theorem to show that a geometric realization $\mathbb{K}$ of $\mathcal{C}(G)$ is homologically trivial. The combinatorial invariant for connected chordal graphs follows from the direct application of the Euler-Poincare formula to $\mathbb{K}$.

The remainder of this paper is organized as follows: The relevant definitions and terminology are summarized in the next section. Required preliminary lemmas are provided in Section 3 and the main results are established in Section 4. A simple illustrative example is presented in Section 5. Closing remarks comprise the final section of this paper.

2. Definitions and Terminology

A *graph* G is a pair $\big(V(G),E(G)\big)$, where $V(G)$ is a finite non-empty set of *vertices* and $E(G)$ is either a set of doubleton subsets of $E(G)$ called *edges* or the empty set $\emptyset$. The *order* of G is the cardinality of $V(G)$ and the *size* of G is the cardinality of $E(G)$. Two vertices $u,v\in V(G)$ are *adjacent* when $e=\{u,v\}\in E(G)$ in which case e is said to *join* u and v. A u-v *walk* is an alternating sequence of vertices and edges beginning with u and ending with v such that every edge joins the vertices immediately preceding and following it. A u-v *path* is a u-v walk in which no vertex is repeated. In this case u is said to be connected to v. G is *connected* if its order is one or if every two vertices in G are connected. A u-v path for which $u=v$ and which contains at least three edges is a *cycle*. The *length* of a cycle is the number of edges contained within it and a *chord* of a cycle is an edge between nonconsecutive vertices in the cycle.

A graph is a *chordal graph* if every cycle of length at least four has a chord. A graph is *complete* if every two of its vertices are adjacent. A graph F is a *subgraph* of G if $V(F)\subseteq V(G)$ and $E(F)\subseteq E(G)$. A *clique* in G is either a vertex or a complete subgraph of G and is *maximal* if it is not a proper subgraph of another clique. The order of a clique is the cardinality of its vertex set and the *clique number* $\omega(G)$ of G is the maximum order among the maximal cliques of G.
A *hypergraph* $\mathcal{H}$ is a pair $(\mathcal{N},\mathcal{E})$, where $\mathcal{N}$ is a finite set of vertices and $\mathcal{E}$ is a set of hyperedges which are non-empty subsets of $\mathcal{N}$ (*it is hereafter assumed that* $\mathcal{H}$ *is reduced, i.e., no hyperedge of* $\mathcal{H}$ *is a subset of another hyperedge of* $\mathcal{H}$). The hypergraph $\mathcal{H}(G)$ associated with graph G has $V(G)$ as its vertices and the hyperedges of $\mathcal{H}(G)$ are the sets of vertices in the maximal cliques of G. $\mathcal{H}(G)$ is connected if G is connected. A connected hypergraph is α*-acyclic* if it has the (nonempty) *running intersection property*, i.e., if there is an ordering $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$ of its hyperedges so that for each $i, 2\le i\le n$, there is a $j_i<i$ such that

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$\emptyset \neq (\varepsilon_i \cap (\cup_{k<i} \varepsilon_k)) \subseteq \varepsilon_{j_i}$ (note the fact that each of these $n-1$ intersections is non-empty implies connectedness) [4, 5].

The *closure* $Cl(S)$ of a finite set S is the family of nonempty subsets of S. The closure $Cl(\mathcal{H})$ of a hypergraph $\mathcal{H}$ is the union of the closure of each of its hyperedges, i.e., $Cl(\mathcal{H}) = \cup_{\varepsilon\in\mathcal{E}} Cl(\varepsilon)$, and the closure $Cl(G)$ of a graph G is $Cl(G) = \cup_{\mathcal{V}\in\mathcal{M}} Cl(\mathcal{V})$, where $\mathcal{V}$ is the set of vertices in a maximal clique of G and $\mathcal{M}$ is the set of all such $\mathcal{V}$'s.

Let $\{a_0, a_1, \cdots, a_k\}$ be a set of geometrically independent points in $\mathbb{R}^n$. The *k-simplex* (or *simplex*) σ^k spanned by $\{a_0, a_1, \cdots, a_k\}$ is the set of points $x \in \mathbb{R}^n$ for which there exist non-negative real numbers $\lambda_0, \lambda_1, \cdots, \lambda_k$ such that $x = \sum_{i=0}^{k} \lambda_i a_i$ and $\sum_{i=0}^{k} \lambda_i = 1$. In this case $\{a_0, a_1, \cdots, a_k\}$ is the *vertex set* of σ^k. A *face* of σ^k is any simplex spanned by a non-empty subset of $\{a_0, a_1, \cdots, a_k\}$. A *finite geometric simplicial complex* (or *complex*) $\mathbb{K}$ is a finite union of simplices such that: (*i*) every face of a simplex of $\mathbb{K}$ is in $\mathbb{K}$; and (*ii*) the non-empty intersection of any two simplices of $\mathbb{K}$ is a common face of each. The *dimension* $dim(\mathbb{K})$ of $\mathbb{K}$ is the largest positive integer *m* such that $\mathbb{K}$ contains an *m*-simplex. The *vertex scheme* of $\mathbb{K}$ is the family of all vertex sets which span the simplices of $\mathbb{K}$. If $\{\mathbb{L}_i : i \in I\}$ is a collection of subcomplexes of $\mathbb{K}$, then $\cup_{i\in I} \mathbb{L}_i$ and $\cap_{i\in I} \mathbb{L}_i \neq \emptyset$ are also subcomplexes of $\mathbb{K}$.

A *finite abstract simplicial complex* (or *abstract complex*) is a finite collection S of finite non-empty sets such that if A is in S, then so is every non-empty subset of A. Thus, the vertex scheme of a complex is an abstract complex - as are finite unions of set closures and finite intersections of set closures. The abstract complex $Cl(G)$ is called the *clique complex* $\mathcal{C}(G)$ of graph G.

Two abstract complexes S and T are isomorphic if there is a bijection φ from the vertex set of Sonto the vertex set of T such that $\{a_0, a_1, \cdots, a_k\} \in S$ if, and only if, $\{\varphi(a_0), \varphi(a_1), \cdots, \varphi(a_k)\} \in T$. Every abstract complex S is isomorphic to the vertex scheme of some geometric simplicial complex $\mathbb{K}$ – in which case $\mathbb{K}$ is the *geometric realization* of S and is uniquely determined (up to linear isomorphism). An isomorphism between S and the vertex scheme of $\mathbb{K}$ is denoted $S \simeq \mathbb{K}$. An *edge* in S is a doubleton subset of vertices contained in S. A distinct pair of vertices u, v of S are *path connected* if there is an alternating sequence $u\{u, x_1\}x_1\{x_1, x_2\}\cdots\{x_n, v\}v$ of vertices and edges of S. The abstract complex S is *connected* when either S has one vertex or all pairs of its vertices are path connected. If S is connected, then so is any geometric realization of S.

To each (simplicial) complex $\mathbb{K}$ there corresponds a chain complex, i.e., abelian groups $\Gamma_p(\mathbb{K})$ and homomorphisms $\partial_{p+1}: \Gamma_{p+1}(\mathbb{K}) \to \Gamma_p(\mathbb{K}), p \geq 0$. If $\mathbb{K}$ is finite and $\xi_p(\mathbb{K})$ is the number of p-simplices in $\mathbb{K}$, then the *rank* of $\Gamma_p(\mathbb{K})$ is $\xi_p(\mathbb{K})$, i.e., $\Gamma_p(\mathbb{K})$ is isomorphic to (denoted " $\cong$ ") the direct sum " $\oplus$ " of $\xi_p(\mathbb{K})$ copies of the additive group of integers $\mathbb{Z}$. The p th *homology group* of $\mathbb{K}$ is the quotient group $H_p(\mathbb{K}) = ker\partial_p / im\partial_{p+1}$ and its *rank* is the p^{th} *Betti number* $b_p(\mathbb{K})$. If $\mathbb{K}$ is connected, then $H_0(\mathbb{K}) \cong \mathbb{Z}$ and $\mathbb{K}$ is *homologically trivial* if it is connected and $H_p(\mathbb{K}) \cong 0, p \geq 1$, where 0 is the trivial group. The complex of a simplex is homologically trivial.

3. Preliminary Lemmas

Several lemmas are required to prove the main results found in the next section. The first three are well known and are repeated here for completeness. The fourth is lesser known and is due to D'Atri *et al.*

Lemma 3.1. [6] (*Euler-Poincaré*) *If* $\mathbb{K}$ *is a complex of finite dimension, then*

$$\sum_{p=0}^{dim(\mathbb{K})} (-1)^p \, \xi_p(\mathbb{K}) = \sum_{p=0}^{dim(\mathbb{K})} (-1)^p \, b_p(\mathbb{K}).$$

Lemma 3.2. [7] (*Mayer-Vietoris*) *Let* $\mathbb{K}$ *be a complex with subcomplexes* $\mathbb{K}_0$ *and* $\mathbb{K}_1$ *such that* $\mathbb{K} = \mathbb{K}_0 \cup \mathbb{K}_1$. *Then there is an exact sequence*

$$\cdots \to H_p(\mathbb{K}_0 \cap \mathbb{K}_1) \to H_p(\mathbb{K}_0) \oplus H_p(\mathbb{K}_1) \to H_p(\mathbb{K}) \to H_{p-1}(\mathbb{K}_0 \cap \mathbb{K}_1) \to \cdots .$$

Lemma 3.3. [8] *Let A be an abelian group, F be a free abelian group, and* $\theta: A \to F$ *be a surmorphism. Then* $A \cong ker\theta \oplus F$.

Lemma 3.4. [9] *If G is a connected chordal graph, then* $\mathcal{H}(G)$ *is α-acyclic.*

The closure operation Cl is also important for proving the main results. The following three lemmas provide the required key properties of Cl. Since the next lemma is straightforward, its proof has been omitted.

Lemma 3.5. *Let* $\{S_i : i \in I\}$ *be a collection of non-empty finite sets. Then the following statements are true:*

(1) $A \subseteq B$ *if, and only if,* $Cl(A) \subseteq Cl(B)$;
(2) $\cap_{i\in I} Cl(S_i) = Cl(\cap_{i\in I} S_i)$;
(3) $\cup_{i\in I} Cl(S_i) \subseteq Cl(\cup_{i\in I} S_i)$;
(4) $A \cap B \neq \emptyset$ *if, and only if,* $Cl(A) \cap Cl(B) \neq \emptyset$; *and*
(5) $A \neq \emptyset$ *if, and only if,* $Cl(A) \neq \emptyset$.

Lemma 3.6. *If the ordering of sets* $(S_1, S_2, \cdots, S_n)$ *has the running intersection property, then the ordering* $(Cl(S_1), Cl(S_2), \cdots, Cl(S_n))$ *also has the running intersection property.*

Proof. The proof results from use of the identity $S_i \cap (\cup_{k<i} S_k) = \cup_{k<i} (S_i \cap S_k)$ in the following

implication chain: $\emptyset \neq S_i \cap (\cup_{k<i} S_k) \subseteq S_{j_i} \Rightarrow \emptyset \neq \cup_{k<i} (S_i \cap S_k) \subseteq S_{j_i} \Rightarrow \emptyset \neq Cl\big(\cup_{k<i} (S_i \cap S_k)\big) \subseteq Cl\big(S_{j_i}\big)$ (Lemma 3.5 *(1),(5)*) $\Rightarrow \emptyset \neq \cup_{k<i} \big(Cl(S_i \cap S_k)\big) \subseteq Cl\big(S_{j_i}\big)$ (Lemma 3.5 *(3)*) $\Rightarrow \emptyset \neq \cup_{k<i} \big(Cl(S_i) \cap Cl(S_k)\big) \subseteq Cl\big(S_{j_i}\big)$ (Lemma 3.5 *(2)*) $\Rightarrow \emptyset \neq Cl(S_i) \cap \big(\cup_{k<i} Cl(S_k)\big) \subseteq Cl\big(S_{j_i}\big)$. Thus, the Cl operation preserves the running intersection property.

Lemma 3.7. *Let the ordering of sets $(S_1, S_2, \cdots, S_n)$ have the running intersection property. If $S' = S_i \cap (\cup_{k<i} S_k)$, then $Cl(S') = Cl(S_i) \cap \big(\cup_{k<i} Cl(S_k)\big)$.*

Proof. Again use the identity $S' = S_i \cap (\cup_{k<i} S_k) = \cup_{k<i} (S_i \cap S_k)$ and let $A \neq \emptyset$. Application of Lemma 3.5 and the definition of Cl provides the following biconditional chain and completes the proof: $A \in Cl(S') \Leftrightarrow A \subseteq S' \Leftrightarrow A \subseteq S_i \cap S_k$ for some $k < i \Leftrightarrow A \in Cl(S_i \cap S_k)$ for some $k < i \Leftrightarrow A \in \cup_{k<i} Cl(S_i \cap S_k) \Leftrightarrow A \in \cup_{k<i} \big(Cl(S_i) \cap Cl(S_k)\big) \Leftrightarrow A \in Cl(S_i) \cap \big(\cup_{k<i} Cl(S_k)\big)$.

4. Main Results

Lemma 4.1. *$Cl\big(\mathcal{H}(G)\big)$ and $\mathcal{C}(G)$ are identical abstract complexes.*

Proof. By definition each hyperedge ε in $\mathcal{H}(G)$ is the vertex set $\mathcal{V}$ of a maximal clique in G so that $Cl(\varepsilon) = Cl(\mathcal{V})$ (Lemma 3.5 (1)). Because of this one-to-one correspondence between hyperedges and maximal cliques, it must therefore be the case that $Cl\big(\mathcal{H}(G)\big) = \cup_{\varepsilon \in \mathcal{E}} Cl(\varepsilon) = \cup_{\mathcal{V} \in \mathcal{M}} Cl(\mathcal{V}) = Cl(G) \equiv \mathcal{C}(G)$.

Corollary 4.2. *$Cl\big(\mathcal{H}(G)\big)$ and $\mathcal{C}(G)$ have the same geometric realizations.*

Proof. Let $\mathbb{K}$ be the geometric realization of $Cl\big(\mathcal{H}(G)\big)$ in which case $Cl\big(\mathcal{H}(G)\big) \simeq \mathbb{K}$. Since $\mathcal{C}(G) = Cl\big(\mathcal{H}(G)\big)$ (Lemma 4.1), it must also be the case that $\mathcal{C}(G) \simeq \mathbb{K}$.

Lemma 4.3. *Let G be a connected chordal graph, $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ be an ordering of the hyperedges of $\mathcal{H}(G)$ which exhibit the running intersection property, and $\mathbb{K}$ be a geometric realization of $\mathcal{C}(G)$. Then:*

(1) there is a subcomplex $\mathbb{K}_i$ of a simplex in $\mathbb{K}$ such that $Cl(\varepsilon_i) \simeq \mathbb{K}_i$;

(2) the subcomplex $\mathbb{L}_i$ of $\mathbb{K}$ such that $\cup_{k<i} Cl(\varepsilon_k) \simeq \mathbb{L}_i$ is connected;

(3) $\mathbb{K}_i \cap \mathbb{L}_i$ is the complex of a simplex of $\mathbb{K}$; and

(4) $\mathbb{K}_i \cup \mathbb{L}_i$ is a connected subcomplex of $\mathbb{K}$.

Proof. Item *(1)* follows from Corollary 4.2; the one-to-one correspondence between maximal cliques in G and hyperedges in $\mathcal{H}(G)$; the fact that for each hyperedge ε_i there is a set $\mathcal{V}$ of maximal clique vertices spanning a simplex in $\mathbb{K}$ such that $\varepsilon_i = \mathcal{V} \Rightarrow Cl(\varepsilon_i) = Cl(\mathcal{V})$ (Lemma 3.5 (1)); and that the abstract complexes $Cl(\varepsilon_i) = Cl(\mathcal{V})$ are therefore both isomorphic to the vertex scheme of the same simplex. Item *(2)* follows from an induction argument on $i \geq 2$ and the running intersection property's nonempty condition $\emptyset \neq \big(Cl(\varepsilon_i) \cap \big(\cup_{k<i} Cl(\varepsilon_k)\big)\big)$, along with the one-to-one correspondence between hyperedges and vertex sets of maximal cliques. Item *(3)*: Since $\mathbb{K}_i$ and $\mathbb{L}_i$ are subcomplexes of $\mathbb{K}$, then so is $\mathbb{K}_i \cap \mathbb{L}_i$. Because $\mathcal{H}(G)$ has the running intersection property, along with Lemma 4.1, Corollary 4.2, and Lemma 3.7, it follows that $\emptyset \neq \big(Cl(\varepsilon_i) \cap \big(\cup_{k<i} Cl(\varepsilon_k)\big)\big) \simeq \mathbb{K}_i \cap \mathbb{L}_i$. But $\big(Cl(\varepsilon_i) \cap \big(\cup_{i<k} Cl(\varepsilon_k)\big)\big) \subseteq Cl\big(\varepsilon_{j_i}\big)$ for some $j_i < i$(Lemma 3.6). Since $Cl\big(\varepsilon_{j_i}\big)$ is isomorphic to the vertex scheme for the complex of a simplex in $\mathbb{K}$, then $\big(Cl(\varepsilon_i) \cap \big(\cup_{i<k} Cl(\varepsilon_k)\big)\big)$ is isomorphic to the vertex scheme of this complex of a simplex or to the complex of one of its faces. In either case, $\mathbb{K}_i \cap \mathbb{L}_i$ is a complex of a simplex of $\mathbb{K}$. Item *(4)*: Since $\mathbb{K}_i$ and $\mathbb{L}_i$ are subcomplexes of $\mathbb{K}$, then so is $\mathbb{K}_i \cup \mathbb{L}_i$. Also, since $\mathbb{K}_i$ and $\mathbb{L}_i$ are connected (items (1) and (2)) and $\mathbb{K}_i$ and $\mathbb{L}_i$ have the complex of a simplex in common (item (3)), then $\mathbb{K}_i \cup \mathbb{L}_i$ must be connected.

Theorem 4.4. *If G is a connected chordal graph and $\mathbb{K}$ is a geometric realization of $\mathcal{C}(G)$, then $\mathbb{K}$ is homologically trivial.*

Proof. Since $\mathbb{K}$ is a geometric realization of $\mathcal{C}(G)$, it is also a geometric realization of $Cl\big(\mathcal{H}(G)\big)$ (Corollary 4.2) and since G is a connected chordal graph, then $\mathcal{H}(G)$ is an α-acyclic hypergraph (Lemma 3.4). Consequently, there is an ordering $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ of the hyperedges of $\mathcal{H}(G)$ which exhibit the running intersection property. Let $\mathbb{K}_i$ and $\mathbb{L}_i, i \in \{2,3,\cdots,n\}$, be subcomplexes of $\mathbb{K}$ such that

$$Cl(\varepsilon_i) \simeq \mathbb{K}_i \text{ and } \cup_{k<i} Cl(\varepsilon_k) \simeq \mathbb{L}_i,$$

in which case $\mathbb{K}_i \cap \mathbb{L}_i$ is the complex of a simplex (Lemma 4.3 *(3)*) and $\mathbb{K}_i \cup \mathbb{L}_i \equiv \mathbb{L}_{i+1}$ is a connected complex (Lemma 4.3 *(4)*). It follows that there is an associated exact sequence (Lemma 3.2)

$$\cdots \to H_p(\mathbb{K}_i \cap \mathbb{L}_i) \to H_p(\mathbb{K}_i) \oplus H_p(\mathbb{L}_i) \to H_p(\mathbb{L}_{i+1}) \to H_{p-1}(\mathbb{K}_i \cap \mathbb{L}_i) \to \cdots.$$

Since $\mathbb{K}_i$ and $\mathbb{K}_i \cap \mathbb{L}_i$ are complexes of simplices (Lemma 4.3 *(1)* and *(3)*), then for $p \geq 1$

$$H_p(\mathbb{K}_i) \cong H_p(\mathbb{K}_i \cap \mathbb{L}_i) \cong 0$$

and for $p = 0$

$$H_0(\mathbb{K}_i) \cong H_0(\mathbb{K}_i \cap \mathbb{L}_i) \cong \mathbb{Z}.$$

Substitution of these group isomorphisms into the above exact sequence induces the following two relevant exact sequences:

$$0 \to H_p(\mathbb{L}_i) \to H_p(\mathbb{L}_{i+1}) \to 0 \quad (p \geq 2)$$

and

$$\cdots \to 0 \to H_1(\mathbb{L}_i) \to H_1(\mathbb{L}_{i+1}) \xrightarrow{\delta} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z} \oplus H_0(\mathbb{L}_i) \xrightarrow{\theta} H_0(\mathbb{L}_{i+1}) \to 0 \to \cdots .$$

Since $i \geq 2$ is arbitrary, the exactness of the first induced sequence provides the isomorphism chain

$$H_p(\mathbb{L}_2) \cong H_p(\mathbb{L}_3) \cong \cdots \cong H_p(\mathbb{L}_{n+1}) = H_p(\mathbb{K}) \quad (p \geq 2) .$$

However, for $p \geq 2$, $H_p(\mathbb{L}_2) \cong 0$ (because $\mathbb{L}_2$ is the complex of a simplex) and it is concluded from the isomorphism chain that

$$H_p(\mathbb{K}) \cong 0 \quad (p \geq 2) .$$

The exactness of the second induced sequence insures that θ is a surmorphism. Also, since $\mathbb{L}_{i+1}$ is connected (Lemma 4.3 *(4)*), then $H_0(\mathbb{L}_{i+1}) \cong \mathbb{Z}$ and it is therefore a free abelian group. It follows from this, the fact that $\mathbb{Z} \oplus H_0(\mathbb{L}_i)$ is abelian, and Lemma 3.3 that

$$\mathbb{Z} \oplus H_0(\mathbb{L}_i) \cong ker\,\theta \oplus H_0(\mathbb{L}_{i+1}) \cong ker\,\theta \oplus \mathbb{Z} .$$

Using the fact that $H_0(\mathbb{L}_i) \cong \mathbb{Z}$ (because $\mathbb{L}_i$ is a connected complex) and $ker\,\theta = im\,\pi$ (from the exact ness of the sequence) in the last expression yields

$$\mathbb{Z} \oplus \mathbb{Z} \cong im\,\pi \ \oplus \mathbb{Z}$$

which implies that $im\,\pi \cong \mathbb{Z}$. Thus, π is a monomorphism so that $ker\,\pi = 0 = im\,\delta$ and exactness at $\mathbb{Z}$ in the second sequence yields the exact sequence

$$0 \to H_1(\mathbb{L}_i) \to H_1(\mathbb{L}_{i+1}) \xrightarrow{\delta} 0$$

which reveals that

$$H_1(\mathbb{L}_i) \cong H_1(\mathbb{L}_{i+1}).$$

Since $i \geq 2$ is arbitrary, the following isomorphism chain holds:

$$H_1(\mathbb{L}_2) \cong H_1(\mathbb{L}_3) \cong \cdots \cong H_1(\mathbb{L}_{n+1}) = H_1(\mathbb{K}) .$$

It can be concluded from this that since $\mathbb{L}_2$ is the complex of a simplex (because $Cl(\varepsilon_1) \simeq \mathbb{L}_2$ and Lemma 4.3 *(1)*), then $H_1(\mathbb{L}_2) \cong 0$ and

$$H_1(\mathbb{K}) \cong 0.$$

Consequently, $\mathbb{K}$ is homologically trivial since it is connected (Lemma 4.3 *(4)*) and $H_p(\mathbb{K}) \cong 0, p \geq 1$.

Theorem 4.5. *If G is a connected chordal graph and $\eta_k(G)$ is the number of cliques of order k in G, then*

$$\sum_{k=1}^{\omega(G)} (-1)^{k-1}\, \eta_k(G) = 1.$$

Proof. Let $\mathbb{K}$ be a geometric realization of the clique complex of G. Since G is connected and chordal, then from Theorem 4.4, $\mathbb{K}$ is homologically trivial. This implies that $b_0(\mathbb{K}) = 1$ and $b_p(\mathbb{K}) = 0$, $p \geq 1$, in which case Lemma 3.1 yields

$$\sum_{p=0}^{dim(\mathbb{K})} (-1)^p\, \xi_p(\mathbb{K}) = 1.$$

The one-to-one correspondence between the $k-1$ dimensional simplices of $\mathbb{K}$ and the cliques of order k in G implies $\xi_{k-1}(\mathbb{K}) = \eta_k(G)$ and $dim(\mathbb{K}) = \omega(G) - 1$. The proof is completed by using these identities in the last expression and resuming over $k, 1 \leq k \leq \omega(G)$.

5. Example

In order to illustrate the theory developed above, consider the connected chordal graph G, its associated hypergraph $\mathcal{H}(G)$, and a geometric realization of $\mathcal{C}(G) = Cl(\mathcal{H}(G))$ shown in Figure 1. By inspection it is seen that $\omega(G) = 4$; the hyperedges of $\mathcal{H}(G)$ are $\varepsilon_1 = \{a, b, c, d\}$, $\varepsilon_2 = \{b, c, e\}, \varepsilon_3 = \{e, f\}$; and $\mathcal{H}(G)$ is α-acyclic since the hyperedge ordering $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ exhibits the (nonempty) running intersection property, i.e.,

$$\varepsilon_3 \cap (\cup_{k<3}\, \varepsilon_k) = \varepsilon_3 \cap (\varepsilon_1 \cup \varepsilon_2) = \{e, f\} \cap \{a, b, c, d, e\} = \{e\} \subset \varepsilon_2$$

and

$$\varepsilon_2 \cap (\cup_{k<2}\, \varepsilon_k) = \varepsilon_2 \cap \varepsilon_1 = \{b, c, e\} \cap \{a, b, c, d\} = \{b, c\} \subset \varepsilon_1 .$$

The closures of these hyperedges are

$$Cl(\varepsilon_1) = \big\{\{a,b,c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a\},\{b\},\{c\},\{d\}\big\},$$

$$Cl(\varepsilon_2) = \big\{\{b,c,e\},\{b,c\},\{b,e\},\{c,e\},\{b\},\{c\},\{e\}\big\},$$

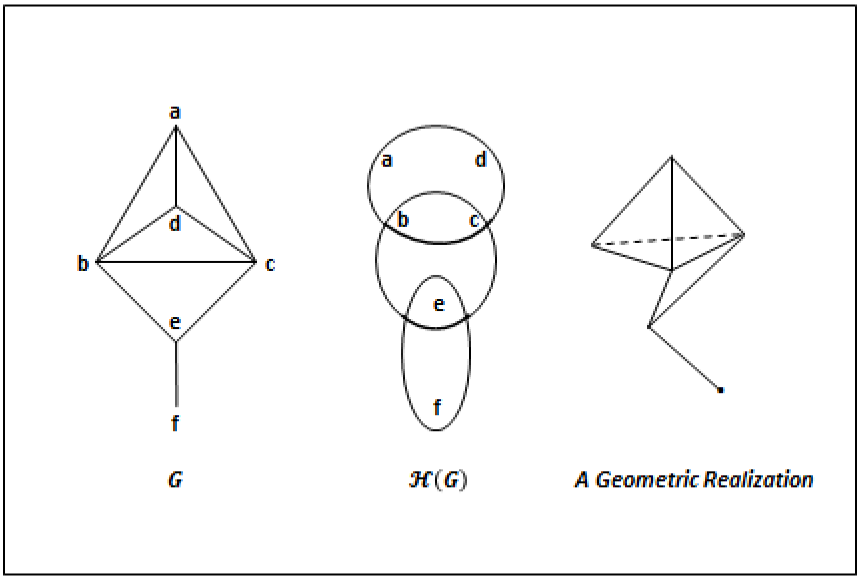
and

$$Cl(\varepsilon_3) = \big\{\{e,f\},\{e\},\{f\}\big\}$$

(these closures are easily seen to validate Lemma 3.6). Since $\mathcal{C}(G) = Cl(\mathcal{H}(G)) = Cl(\varepsilon_1) \cup Cl(\varepsilon_2) \cup Cl(\varepsilon_3)$, then the clique complex of G is the set

$$\mathcal{C}(G) = \left\{ \begin{matrix} \{a,b,c,d\},\{a,b,c\},\{a,b,d\}\{a,c,d\},\{b,c,d\},\{b,c,e\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{b,e\}, \\ \{c,e\},\{e,f\},\{a\},\{b\},\{c\},\{d\},\{e\},\{f\} \end{matrix} \right\}.$$

Figure 1. A chordal graph G, its associated hypergraph $\mathcal{H}(G)$, and a geometric realization of $\mathcal{C}(G)$.



Theorem 4.4 is validated by observing that since it contains no "holes", the geometric realization of $\mathcal{C}(G)$ in Figure 1 is homologically trivial (or somewhat more formally, there are no linear combinations of k-simplices in $\mathcal{C}(G)$ which form k-cycles that enclose "holes"). Theorem 4.5 is validated since by inspection it is easily determined that $\eta_1(G) = 6$, $\eta_2(G) = 9$, $\eta_3(G) = 5$, and $\eta_4(G) = 1$ in which case

$$\sum_{k=1}^{4}(-1)^{k-1}\eta_k(G) = 6 - 9 + 5 - 1 = 1\,.$$

6. Concluding Remarks

It has been shown that any connected chordal graph has a clique complex that is isomorphic to the vertex scheme of (up to linear isomorphism) a unique homologically trivial geometric realization. As a consequence of this, the Euler-Poincaré formula provides a combinatorial invariant based simply upon the number of cliques in the graph and whose unit value holds for all connected chordal graphs. Accordingly, the contrapositive version of Theorem 4.5 serves to determine when a graph is not a connected chordal graph.

It is also interesting to note that since a tree is a connected chordal graph, then Theorem 4.5 subsumes the well-known fact that if a graph is a tree, then the difference between its order and its size is one (more specifically, if G is a tree, then $\omega(G) = 2$, $\eta_1(G)$ is its order, and $\eta_2(G) = \eta_1(G) - 1$ is its size so that $\sum_{k=1}^{\omega(G)}(-1)^{k-1}\eta_k(G) = \eta_1(G) - \eta_2(G) = \eta_1(G) - (\eta_1(G) - 1) = 1$).

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