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# Peculiar Dynamics of Phase-space Embedded SU(2) Hamiltonians

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**ABSTRACT:** The Maximum Entropy Principle is used to analyze a phase-space embedded SU(2) Hamiltonian that exhibits a very complex dynamics. It is seen that the uncertainty principle becomes an invariant of the motion.

**KEYWORDS:** Non-linear Semi-quantum Dynamics, Maximum Entropy Principle Approach, Complex Dynamics

## 1. INTRODUCTION

The extreme complexity of phase space's trajectories that are very sensitive to small changes in the initial conditions is the signature of classical chaos, accompanied by i) an ostensibly random allotment of phase points on a Poincare's surface of section, and ii) an exponentially rapid separation of two initially close trajectories [1]. Instead, it is clear that the state vector of a closed quantum system cannot exhibit chaotic motion in Hilbert space. However, the interaction between a quantum system and a classical one may instead lead to authentic chaotic motion of the quantum component, a phenomenon known as semi-quantum chaos [2]. Remark that the term semi-quantum is reserved to systems for which neither the quantum part nor the classical part would be chaotic by themselves. As Ballentine has pointed out [3], preferring the term semi-quantum over semi-classical is justified because the latter is restricted to scenarios in which the Feynman sum over paths is approximated by a sum over only the classical paths, or alternatively, as a WKB approximation to the wave function. Semi-quantum condition, instead, implies that one part is treated classically and the other one in quantum fashion.

In the wake of Ballentine's work, we consider here the Maximum Entropy Principle Approach's (MEP) environment to deal with a kind of SU(2) non-linear semi quantum Hamiltonians: those which i) do not explicitly depend on time and ii) their quantum subsystems close a partial Lie algebra under commutation with the Hamiltonian [4,5,6]. We will show that for such MEP environment i) when a Lie algebra can be associated to a semi quantum non-linear Hamiltonian through the closure condition [4], it is possible to integrate the quantum degrees of freedom of this semi quantum non-linear system and ii) for the SU(2) Lie algebra case, this semi quantum non-linear dynamics exhibits some invariants which are very helpful to study the transition from regular to irregular dynamics. One of these invariants is the generalized uncertainty principle (GUP) of refs. [5,7], which also arises in semi quantum dynamics and prevents the system from turning into a "mathematical artifact" (paraphrasing ref. [3]) given that its constant value is fixed through the initial conditions. Accordingly, is it possible to assimilate the ensuing irregular SU(2) non-linear dynamics to semi-quantum chaos? Or are we dealing with a different type complex dynamics? Answering this question is our present leitmotiv.

The paper is organized as follows: some preliminary considerations are the subject of Section II. Sects. III and IV are devoted to explain how the MEP approach tackles the dynamics of semi-quantum systems. Sec. V is devoted to two illustrative Hamiltonian-examples. In Section VI we present numerical simulations and some conclusions are drawn in Sect. VII.

## 2. PRELIMINARIES

Consider a system that possesses both quantum and classic degrees of freedom, with a coupling amongst them, that we call semi-quantum [3,8,9,10,11,12]. The associated Hamiltonian is of the general form [13]

$$\hat{H} = \hat{H}_q + \hat{H}_{cl} + \hat{H}_{\text{int}} \quad (1)$$

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where $\hat{H}_q$, $\hat{H}_{cl}$, and $\hat{H}_{\text{int}}$ are the quantum, classical, and interaction parts, respectively. There exist many situations in which a semi-quantum description has been attempted [8,11,14]. Indeed, many semi-quantum Hamiltonians are found in the literature [3,6,15,16,17,18,19,20,21,22] and M. A. Porter [8] made an exhaustive compilation of physical systems for which this kind of description is relevant. One may highlight vibrating quantum billiards as a useful abstraction of the ensuing semi-quantum dynamics [23].

In this work we will provide a Maximum Entropy Principle (MEP) vantage point for these systems, that employs dynamic invariants in order to establish their main features. These invariants will allow us to i) adequately define initial conditions and ii) follow the details of the temporal evolution, from regular regimes to its irregular ones. Our main invariant, as we previously stated, is none other than the generalized uncertainty principle (GUP) [7]. From a MEP viewpoint, in the analysis of a semi-quantum dynamics one deals with a peculiar dynamic "working" space spanned by the variables $\langle \hat{O}_1 \rangle, ..., \langle \hat{O}_N \rangle, q_1, ..., q_n, p_1, ..., p_n$. The first $N$ $\langle \hat{O}_1 \rangle, ..., \langle \hat{O}_N \rangle$ variables are the mean values of a set of non-commuting observable that close a partial Lie semi-algebra under commutation with the Hamiltonian. These will be regarded as our "constraints". The $q_1, ..., q_n, p_1, ..., p_n$ are $2n$ classical variables of the system. As in the full quantum case developed in ref. [4], a new semi quantum closure condition defines an $N\times N$ dynamic matrix $G(q_i, p_i)$ which governs the dynamics of the system's quantum degrees of freedom. This matrix is now of a semi-quantum nature since it depends upon classical degrees of freedom. We will define a sufficient condition that the $G(q_i, p_i)$ matrix must fulfill in order the GUP constitute a dynamic invariant. For the dynamic evolution of the classical ingredients we follow the prescription given in refs. [5,6,11,13,14,19,20], i.e. the energy of the system is taken to coincide with the expectation value of the Hamiltonian, $\langle \hat{H} \rangle = Tr(\hat{\rho}\hat{H})$, traced over the quantum state, $\hat{\rho}$. In turn, $\langle \hat{H} \rangle$ will generate the temporal evolution of the classical degrees of freedom in the orthodox Classical Mechanics' fashion. The MEP's point of view approach to semi quantum non-linear systems takes advantage of four facts:

1. It is possible to alternatively describe the time-evolution of the quantum degrees of freedom also in the dual space of Lagrange multipliers associated to the quantum observable [4,24].
2. If it is possible to associate to the semi quantum non-linear system a Lie algebra under commutation operation with the Hamiltonian, then it is always feasible to find a statistical operator $\hat{\rho}(t)$ of maximum entropy, for all times.
3. f it is possible to associate to the semi quantum non-linear system a Lie algebra under commutation operation with the Hamiltonian, then it is possible to integrate the quantum degrees of freedom of the non-linear system.
4. The existence of the GUP-invariant, $I^H$ say, for the SU(2) Lie algebra makes it possible to analyze the dynamics of the system in different regimes (irregular and regular) just by varying $I^H$'s value.

**3. THE MAXIMUM ENTROPY FORMALISM FOR NON-LINEAR SEMI-QUANTUM SYSTEMS**

Consider the Hamiltonian given by Eq. (1). The classical degrees of freedom are the canonical conjugate variables $(q_i, p_i)$. We choose for the Hamiltonian (1) the specific form

$$\hat{H} = \sum_j \sum_{i=1}^{n} a_j(q_i, p_i)\hat{O}_j \sum_{i=1}^{n} F(q_i, p_i) \quad (2)$$

where the first term includes both the $\hat{H}_q$ and $\hat{H}_{\text{int}}$ ingredients, the $\hat{O}_j$'s being quantum operators, while the last term is a purely classical one, with $F(q_i, p_i)$ functions of the canonically conjugate classical variables $(q_i, p_i)$. We have in fact a family of Hamiltonians with a classical phase-space substratum. The first term, is a linear superposition of quantum operators (belonging to a particular Lie algebra) which closes a partial Lie algebra under commutation with Eq. (1) through the semi quantum closure condition [5]

$$\left[\hat{H}(t), \hat{O}_j\right] = i\hbar \sum_{r=0}^{N} \sum_{i=1}^{n} g_{rj}(q_i, p_i)\hat{O}_r \quad (3)$$

the $g_{rj}(q_i, p_i)$ are the coefficients of a $N\times N$ matrix $G(q_i, p_i)$, whose nature is semi-quantum, given that the $a_j(q_i, p_i)$ terms in Eq. (2) may contain the classical degrees of freedom $q_i, p_i$. The MEP formalism [4,24] deals with the quantum degrees of freedom of the system and provides a density operator of maximum entropy for the initial time $\hat{\rho}(t_0)$ (the boundary condition) which has the form [4]

$$\hat{\rho}(t_0) = \exp\left(-\lambda_0 \hat{I} - \sum_{j=1}^{N} \lambda_j \hat{O}_j\right) \quad (4)$$

and is expressed in terms of *N+1* Lagrange multipliers $\lambda_1, \lambda_2, ..., \lambda_N$. According to Jayne's Information Theory [25,26], the statistical operator $\hat{\rho}(t_0)$ is constructed starting from the knowledge of the expectation values of *N+1* operators $\hat{O}_j$ termed as the constraints

$$\langle \hat{O}_j \rangle = Tr(\hat{\rho}(t)\hat{O}_j); \quad j = 0,1,..., N \quad (5)$$

where the subindex 0 refers to the normalization condition

$$Tr(\hat{\rho}) = 1 \quad (6)$$

given that the identity operator $\hat{O}_0 = \hat{I}$ must to be included in order to fulfill condition (6). As it was established by Alhassid & Levine [4], the constraints must be linearly independent but not necessarily commuting ones. $\lambda_0$ is the one associated to the identity operator *I* and is obtained as

$$\lambda_0 = \ln\left\{Tr\left[\exp\left(-\sum_{j=1}^{N}\lambda_j \hat{O}_j\right)\right]\right\} \quad (7)$$

It is possible to demonstrate [5] that the above formalism applies also to for semi quantum instances. In order for the statistical operator (4) to preserve at all times the form it has at $t = t_0$ (corresponding to a state of maximum entropy), the semi quantum closure condition (3) must hold. In this way, the equation of motion of the density operator [4]

$$\frac{\partial\hat{\rho}(t)}{\partial t} = \frac{1}{i\hbar}\left[\hat{H}(t), \hat{\rho}(t)\right] \quad (8)$$

may be converted into a set of coupled non-linear equations for the Lagrange multipliers in straightforward fashion, as Alhassid & Levine exemplified in ref. [4] for the full quantum case (see ref. [5] for more details). One obtains

$$\frac{\partial\lambda_r}{\partial t} = \sum_{j=0}^{N}\sum_{i=1}^{n} g_{rj}(q_i, p_i)\lambda_j; \quad r = 1,..., N \quad (9)$$

Thus, for the semi-quantum non-linear case we have obtained a density operator $\hat{\rho}(t)$ of maximum entropy of the kind (4), for all times, at the cost of having to deal with a set of non-linear coupled equations of motion for the Lagrange parameters. Since the statistical operator obeys Eq. (8), the entropy $S = -Tr(\hat{\rho}\ln\hat{\rho})$ is a constant of the motion [4], i.e.

$$S(t_0) = S(t) \quad (10)$$

To derive the time evolution of the expectation values of the constraints generated by Eq. (3), we can proceed in an entirely similar fashion (Cf. ref. [24] for the full quantum case). One gets the following non-linear set of coupled equations of motion for the quantum degrees of freedom of the semi quantum system (2)

$$\frac{d\langle \hat{O}_k \rangle}{dt} = -\sum_{r=1}^{N}\sum_{i=1}^{n} g_{rk}(q_i, p_i)\langle \hat{O}_r \rangle; \; k = 1,..., N$$

(11)

which are the generators of a Lie algebra and are obtained through Eq. (3). As we were able to close the algebra (see Eq. (3)) we can obtain the mean values of the quantum degrees of freedom in the fashion

$$\langle \hat{O}_j \rangle(t) = Tr\left\{\hat{\rho}(t)\hat{O}_j\right\}; \; j = 1,..., N \quad (12)$$

and this means we have integrated the quantum degrees of freedom of the non-linear semi-quantum system given by Eq. (2). In fact, Eq. (12) is a primitive of the equation of motion (11). Thus, if we take the time derivative of Eq. (12), we obtain

$$\frac{d\langle \hat{O}_j \rangle}{dt} = \frac{d}{dt}Tr\left\{\hat{\rho}(t)\hat{O}_j\right\} = Tr\left\{\frac{d}{dt}\hat{\rho}(t)\hat{O}_j\right\}; \; j = 1,..., N \quad (13)$$

As in the Schrödinger representation the quantum operators do not depend explicitly on the time, all the time dependence is contained in the MEP density operator $\hat{\rho}(t)$ through the time dependence of the Lagrange parameters $\left(\frac{\partial\hat{O}_j}{\partial t} = 0\right)$. Then, from Eq. (13) we obtain (see also Eq. (8))

$$\frac{d\langle \hat{O}_j \rangle}{dt} = Tr\left\{\frac{d\hat{\rho}(t)}{dt}\hat{O}_j\right\} = Tr\left\{\frac{1}{i\hbar}\left[\hat{H}(t), \hat{\rho}(t)\right]\hat{O}_j\right\}; \; j = 1,..., N \quad (14)$$

and one take advantage of the invariance of the trace under commutation operations

$$\frac{d\langle \hat{O}_j \rangle}{dt} = -Tr\left\{\frac{1}{i\hbar}\hat{\rho}(t)\left[\hat{H}(t), \hat{O}_j\right]\right\}; \; j = 1,..., N \quad (15)$$

Finally, taking into account the semi-quantum closure condition (3), Eq. (15) may be cast as

$$\frac{d\langle\hat{O}_j\rangle}{dt}=-Tr\left\{\hat{\rho}(t)\sum_{r=0}^{N}\sum_{i=1}^{n}g_{rj}(q_i,p_i)\hat{O}_r\right\}$$
$$=-\sum_{r=0}^{N}\sum_{i=1}^{n}g_{rj}(q_i,p_i)\langle\hat{O}_r\rangle(t);\ j=1,..,N \quad (16)$$

which is the generalized Ehrenfest theorem given by Eq. (11). *Summing up, if we are able to close a semi Lie algebra under commutation with the semi-quantum non-linear Hamiltonian* (2), *then we will be able to integrate the equations of motion of the quantum degrees of freedom, in spite of the non-linearity accrued to the terms* $a_j(q_i,p_i)$. Eqs. (12) may also be obtained via

$$\langle\hat{O}_j\rangle(t)=-\frac{\partial\lambda_0}{\partial\lambda_j};\ j=1,...,N \quad (17)$$

As the quantum subsystem of the non-linear semi quantum Hamiltonian (2) is a linear superposition of the generators of the Lie algebra, linked via the closure condition (3), we will also be able to obtain another constant of the motion: the mean value of the non-linear Hamiltonian (2). Thus, the density operator of maximum entropy may be used to calculate the mean value of the Hamiltonian (2) in the fashion

$$\langle\hat{H}\rangle=Tr(\hat{\rho}\hat{H})=\sum_{j}\sum_{i=1}^{n}a_j(q_i,p_i)\langle\hat{O}_j\rangle+\sum_{k}\sum_{i=1}^{n}F_k(q_i,p_i) \quad (18)$$

The entropy at the maximum acquires the form [4]

$$S(\hat{\rho})=-Tr\{\hat{\rho}(t)\ln\hat{\rho}(t)\}=\lambda_0+\sum_{j=1}^{N}\lambda_j\hat{O}_j \quad (19)$$

and is a constant of the motion.

Concerning the system's classical degrees of freedom, the energy is taken to coincide with the quantum expectation value of the semi-quantum Hamiltonian [11,13,14,19,20] given by Eq. (18) and the temporal evolution of the classical variables is determined via [5,13]

$$\frac{dq_i}{dt}=\frac{\partial\langle\hat{H}\rangle}{\partial p_i};\ i=1,...,n \quad (20)$$

$$\frac{dp_i}{dt}=-\frac{\partial\langle\hat{H}\rangle}{\partial q_i};\ i=1,...,n \quad (21)$$

Although Eqs. (20) and (21) generate the dynamic evolution of the classical variables, its nature is semi-quantum as well because it gets entangled with the mean values of the quantum variables. Thus, the interplay between classical and quantum variables acquires special relevance through the matrix $G(q_i,p)$ and the energy $\langle\hat{H}\rangle=\langle\hat{H}\rangle(q_i,p_i,\langle\hat{O}_j\rangle)$, since they are the keys to determining the entanglement between the quantum and classical degrees of freedom of a semi-quantum system of the type (2).

### 3.1 GENERALIZED UNCERTAINTY PRINCIPLE FOR SEMI-QUANTUM DYNAMICS

Now we appeal to the general expression for the uncertainty principle, valid for any two non-commuting operators $[\hat{A},\hat{B}]=i\hat{C}$, given in ref. [27]

$$(\Delta\hat{A})^2(\Delta\hat{B})^2-\left[\frac{1}{2}\langle\hat{A}\hat{B}+\hat{B}\hat{A}\rangle-\langle\hat{A}\rangle\langle\hat{B}\rangle\right]^2\geq\frac{1}{4}\langle C\rangle^2 \quad (22)$$

For any pair of operators belonging to the Lie algebra generated by the semi quantum closure condition (3) we have

$$(\Delta\hat{O}_i)^2(\Delta\hat{O}_j)^2-\left[\frac{1}{2}\langle\hat{O}_i\hat{O}_j+\hat{O}_j\hat{O}_i\rangle-\langle\hat{O}_i\rangle\langle\hat{O}_j\rangle\right]^2\geq-\frac{1}{4}\langle[\hat{O}_i,\hat{O}_j]\rangle^2 \quad (23)$$

Keeping in mind that, via Eq. (3), we are able to find a complete set of N non-commuting observable (CSNCO), which are our quantum degrees of freedom, we define the following expression

$$I^H=\sum_{i=1}^{N}\sum_{\substack{j=1\\i<j}}^{N}\left\{(\Delta\hat{O}_i)^2(\Delta\hat{O}_j)^2-\left[\frac{1}{2}\langle\hat{O}_i\hat{O}_j+\hat{O}_j\hat{O}_i\rangle-\langle\hat{O}_i\rangle\langle\hat{O}_j\rangle\right]^2\right\}\geq$$
$$\geq-\sum_{i=1}^{N}\sum_{\substack{j=1\\i<j}}^{N}\frac{1}{4}\langle[\hat{O}_i,\hat{O}_j]\rangle^2. \quad (24)$$

Eq. (24) is a sum over all the possible pairs of operators entering the CSNCO obtained through Eq. (3), which we have called the generalized uncertainty principle (see ref. [7]).

### 3.2 INVARIANTS OF THE MOTION

In this Section we discuss the several motion invariants that, of course, considerably constrain the evolution dynamics of our system and gives raise to their intrinsic peculiarities. We begin by defining the quantum correlation coefficients following ref. [27]

$$K_{ij}(t) = \frac{1}{2}\left\langle \hat{O}_i\hat{O}_j + \hat{O}_j\hat{O}_i \right\rangle_{(t)} - \left\langle \hat{O}_i \right\rangle_{(t)} \left\langle \hat{O}_j \right\rangle_{(t)} \quad (25)$$

which are the components of the positive definite quantum correlation matrix *K(t)*, corresponding to the generators of the Lie algebra according to Eq. (3). One appreciates that the left hand side of Eq. (24) may be obtained as the sum over the principal minors of order 2 of this correlation matrix. If we define operator $\hat{L}_{jk} = \frac{1}{2}\left(\hat{O}_i\hat{O}_j + \hat{O}_j\hat{O}_i\right)$, it is easy to see that i) $\hat{L}_{jk} = \hat{L}_{kj}$ and ii) $\hat{L}_{jj} = \left(\hat{O}_j\right)^2$. With the help of Eqs. (3) and (11) we are led to

$$\frac{d\left\langle \hat{L}_{jk} \right\rangle}{dt} = -\sum_{r=1}^{N}\sum_{i=1}^{n}\left\{ g_{rk}(q_i,p_i)\left\langle \hat{L}_{jr} \right\rangle + g_{rj}(q_i,p_i)\left\langle \hat{L}_{kr} \right\rangle \right\} \quad (26)$$

Now, if we take the time derivative on the left hand side of Eq. (24) and use Eq (26), we find that, in order for $I^H$ to be a dynamic invariant, it is necessary that

- $\sum_{i=1}^{n} g_{jj}(q_i,p_i) = 0 \Rightarrow g_{jj}(q_i,p_i) = 0; i = 1,\ldots,n$, and
- $$\sum_{i=1}^{n}\left\{ g_{rj}(q_i,p_i) + g_{jr}(q_i,p_i)\right\} = 0 \Rightarrow \sum_{i=1}^{n} g_{rj}(q_i,p_i) = -\sum_{i=1}^{n} g_{jr}(q_i,p_i) \Rightarrow$$ $$\Rightarrow g_{rj}\sum_{i=1}^{n} g_{rj}(q_i,p_i) = -g_{jr}\sum_{i=1}^{n} g_{rj}(q_i,p_i); i = 1,\ldots,n$$

Matrix $G(q_i,p_i)$ becomes then anti-symmetric and the generalized uncertainty principle (24) becomes indeed a constant of the motion. It is also possible to demonstrate that if the matrix $G(q_i,p_i)$ is an anti-symmetric one, all the principal minors of order *r = 1;2;...;N* belonging to the correlation matrix are invariants of the motion too (see ref. [28] for more details). Accordingly, the number of invariants of motion equals that of quantum degrees of freedom.

### 3.3 CONSEQUENCES OF THE ALGEBRA CLOSURE

The first consequence of fulfilling Eq. (3) is to obtain a statistical operator of maximum entropy $\hat{\rho}(t)$ for the semi-quantum system (2) given by Eq. (4). This operator is valid for any time t due to the semi-quantum closure condition (3). It guarantees the fact that the surprisal $-\ln\hat{\rho}(t) = \sum_{r=0}^{N}\lambda_r(t)\hat{O}_r$ is an exact solution of the Liouville equation (8). This enables one to turn this Eq. (8) into a set of *N* coupled non-linear differential equations for the Lagrange parameters (see Eqs. (9)). Additionally, this statistical operator makes it possible to obtain the mean values of the N relevant operators generated through Eq. (3), in the fashion $\left\langle \hat{O}_j \right\rangle(t) = Tr\left(\hat{\rho}(t)\hat{O}_j\right)$, which, in virtue of the closure condition (3), becomes a primitive of Eq. (11). This means that we have integrated the quantum degrees of freedom of a non-linear semi-quantum system (2). We also can integrate the mean value of the Hamiltonian (2) (see Eq. (18)).

As a consequence, the entropy $S(\hat{\rho}) = -Tr(\hat{\rho}\ln\hat{\rho})$ remains an invariant of the motion for the semi-quantum case too. It is expressed in terms of both the set of relevant operators and their associated Lagrange multipliers (see Eq. (19)). The main difference between the full quantum case and the semi-quantum case is that, for the latter, not only the mean values but also the Lagrange parameters obey non-linear equations of motion (see Eqs. (9) and (11)).

The closure condition enables one to obtain the generalized uncertainty principle for the quantum degrees of freedom of the semi quantum system (see Eq. (24)). It is an invariant of the motion if the semi-quantum matrix $G(q_i,p_i)$, generated through the closure condition, is anti-symmetric.

### 3.4 THE SU(2) INSTANCE

In what circumstances does the dynamic matrix $G(q_i,p_i)$ become anti-symmetric? The answer depends upon the Lie algebra associated to the system. It is well-known that $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ is a basis of the SU(2) algebra. The pertinent commutation relationships [29] hold

$$\left[\hat{\sigma}_j, \hat{\sigma}_k\right] = 2i\ \varepsilon_{jkl}\ \hat{\sigma}^l \quad (27)$$

and the semi-quantum Hamiltonian of the type (2) becomes

$$\hat{H} = \sum_{j=1}^{3}\sum_{i=1}^{n} a_j(q_i, p_i)\hat{\sigma}_j + \frac{p^2}{2m} + V(q) \quad (28)$$

where $\hat{\sigma}_j$ are the generators of SU(2).

**Proposition:** If a set of operators, which fulfills the commutation relation (27conmut), closes a commutation algebra with a Hamiltonian of the type (28), then the semi-quantum matrix $G(q_i, p_i)$ of the system, defined by means of the closure condition (3), is an anti-symmetric one.

**Proof:** If $\hat{\sigma}_k$ belongs to the relevant set, then:

$$\left[\hat{H}, \hat{\sigma}_k\right] = 2i\sum_{j=1}^{3} a_j(q_i, p_i)\left[\hat{\sigma}_j, \hat{\sigma}_k\right] = i\sum_{j=1}^{3}\sum_{l=1}^{3}\sum_{i=1}^{n} a_j(q_i, p_i)\varepsilon_{jkl}\hat{\sigma}_l$$

. Taking into account the closure condition (3):

$$\sum_{l=1}^{3}\left\{\sum_{i=1}^{n} g_{lk}(q_i, p_i) - \sum_{i=1}^{n}\sum_{j=1}^{3} 2a_j(q_i, p_i)\varepsilon_{jkl}\right\}\hat{\sigma}_l = 0$$

. Now, as the operators $\hat{\sigma}_l$ are linearly independent,

$$\sum_{i=1}^{n}\left\{g_{lk}(q_i, p_i) - \sum_{j=1}^{3} 2a_j(q_i, p_i)\varepsilon_{jkl}\right\} = 0.$$

The index $i$ runs over the classical variables $(q_i, p_i)$, which are, of course, linearly independent. If we consider a fixed $i$ value, for any element $g_{lk}(q_i, p_i)$ belonging to $G(q_i, p_i)$ it is true that

$$g_{lk}(q_i, p_i) = \sum_{j=1}^{3} 2a_j(q_i, p_i)\varepsilon_{jkl}\ ; \quad \forall\, i = 1,\ldots,n.$$

Accordingly, $g_{ll}(q_i, p_i) = 0$, and $g_{lk}(q_i, p_i) = -g_{kl}(q_i, p_i)$, $\forall$ k,l $\Rightarrow G(q_i, p_i)$ is anti-symmetric. *End of Proof*

Every Hamiltonian that closes an algebra with the SU(2) generators is accompanied by the invariant (24) expressed in the fashion [7].

$$I^H = 3 - 2\left[\langle\hat{\sigma}_x\rangle^2 + \langle\hat{\sigma}_y\rangle^2 + \langle\hat{\sigma}_z\rangle^2\right] = 3 - 2\langle\hat{\sigma}\rangle^2. \quad (29)$$

Because of the uncertainty principle (see Eq. (24)) $I^H \geq -\frac{1}{4}\sum_{\substack{j,k=1 \\ j<k}}^{3}\left\langle\left[\hat{\sigma}_j, \hat{\sigma}_k\right]\right\rangle^2$, and Schwarz' inequality entails

$\langle\hat{\sigma}\rangle^2 < 1$, i.e., the uncertainty principle for the SU(2) Lie algebra, that can be expressed in the guise

$$0 < \langle\hat{\sigma}_x\rangle^2 + \langle\hat{\sigma}_y\rangle^2 + \langle\hat{\sigma}_z\rangle^2 < 1 \quad (30)$$

defining the celebrated Bloch sphere of the system.

## 4. ILUSTRATION: A SPECIAL SU(2) HAMILTONIAN

Let us consider the following specific Hamiltonian [3]

$$\hat{H} = B\hat{\sigma}_z + Cq\hat{\sigma}_x + \frac{p^2}{2m} + \frac{q^4}{4} \quad (31)$$

where $q$ and $p$ are canonically conjugate classical variables and $\hat{\sigma}_i$ are spin(1/2)-operators. The $B\hat{\sigma}_z$ term is the spin Hamiltonian, $\frac{p^2}{2m} + \frac{q^4}{4}$ yields the classical Hamiltonian, and $Cq\hat{\sigma}_x$ represents the interaction between them. By considering the Lie algebra $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ as the relevant set, via Eq. (3) one gets a set of non-commuting observable whose mean values are the quantum degrees of freedom of the system, while q and p are the classical ones. Eq. (3) leads to the following *G(q)* anti-symmetric matrix

$$G(q) = \begin{pmatrix} 0 & -2B & 0 \\ 2B & 0 & -2qC \\ 0 & 2qC & 0 \end{pmatrix} \quad (32)$$

Eqs. (9) and (11) yield both the quantum equations of motion and those of the Lagrange multipliers associated to them

$$\frac{d\langle\hat{\sigma}_x\rangle}{dt} = -2B\langle\hat{\sigma}_y\rangle \quad (33)$$

$$\frac{d\langle\hat{\sigma}_y\rangle}{dt} = 2B\langle\hat{\sigma}_x\rangle - 2Cq\langle\hat{\sigma}_z\rangle \quad (34)$$

$$\frac{d\langle\hat{\sigma}_z\rangle}{dt} = 2Cq\langle\hat{\sigma}_y\rangle \quad (35)$$

$$\frac{d\lambda_x}{dt} = -2B\lambda_y \quad (36)$$

$$\frac{d\lambda_y}{dt} = 2B\lambda_x - 2Cq\lambda_z \quad (37)$$

$$\frac{d\lambda_x}{dt} = 2Cq\lambda_y \quad (38)$$

Moreover, the quantum state of the system can be described by means of the state operator [30]

$$\hat{\rho} = \exp\left(-\lambda_0 - \lambda_x\hat{\sigma}_x - \lambda_y\hat{\sigma}_y - \lambda_z\hat{\sigma}_z\right) = \frac{1}{2}\left(\hat{I} + \frac{\tanh|\vec{\alpha}|}{|\vec{\alpha}|}\vec{\alpha}.\vec{\sigma}\right) \quad (39)$$

with $\vec{\sigma} = \left(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\right)$, $|\vec{\alpha}| = \sqrt{\lambda_x^{\,2} + \lambda_y^{\,2} + \lambda_x^{\,2}}$, and $\lambda_0$, which derives from the normalization condition, is $\lambda_0 = \ln\left[2\cosh\left(\sqrt{\lambda_x^{\,2} + \lambda_y^{\,2} + \lambda_x^{\,2}}\right)\right]$. The state operator $\hat{\rho}(t)$ enables us to evaluate (to integrate) the mean values of the quantum degrees of freedom

$$\left\langle\hat{\sigma}_x\right\rangle = -\frac{\tanh|\vec{\alpha}|}{|\vec{\alpha}|}\lambda_x \quad (40)$$

$$\left\langle\hat{\sigma}_y\right\rangle = -\frac{\tanh|\vec{\alpha}|}{|\vec{\alpha}|}\lambda_y \quad (41)$$

$$\left\langle\hat{\sigma}_z\right\rangle = -\frac{\tanh|\vec{\alpha}|}{|\vec{\alpha}|}\lambda_z \quad (42)$$

Taking time derivatives in these equations, and minding Eqs. (36) to (38), we straightforwardly recover the equations of motion (33), (34), and (35), as the MEP approach prescribes. Eqs. (40), (41) and (42) show clearly that, if it is possible to associate to a non-linear Hamiltonian (of the type of Eq. (2)) a Lie algebra (under commutation operation through Eq. (3)), *then it is possible to integrate the quantum degrees of freedom of the semi quantum non-linear system*. The mean value of the Hamiltonian (31) becomes

$$\left\langle\hat{H}\right\rangle = B\left\langle\hat{\sigma}_z\right\rangle + Cq\left\langle\hat{\sigma}_x\right\rangle + \frac{p^2}{2m} + \frac{q^4}{4} \quad (43)$$

which, together with Eqs. (20) and (21) leads to the classical equations of motion

$$\frac{dq}{dt} = \frac{p}{m} \quad (44)$$

$$\frac{dp}{dt} = -C\left\langle\hat{\sigma}_x\right\rangle - q^3. \quad (45)$$

The quantum correlation matrix's components are [5]:

$K_{xx} = 1 - \left\langle\hat{\sigma}_x\right\rangle^2$, $K_{yy} = 1 - \left\langle\hat{\sigma}_y\right\rangle^2$, $K_{zz} = 1 - \left\langle\hat{\sigma}_z\right\rangle^2$, $K_{xy} = K_{yx} = -\left\langle\hat{\sigma}_x\right\rangle\left\langle\hat{\sigma}_y\right\rangle$, $K_{xz} = K_{zx} = -\left\langle\hat{\sigma}_x\right\rangle\left\langle\hat{\sigma}_z\right\rangle$, $K_{yz} = K_{zy} = -\left\langle\hat{\sigma}_y\right\rangle\left\langle\hat{\sigma}_z\right\rangle$ As the matrix *G(q)*, given by Eq. (32), is anti-symmetric, three dynamic invariants arise from the quantum correlation matrix's components: the sum over the principal minors of order 1, 2, and 3, respectively. They are

$$Tr(K) = K_{xx} + K_{yy} + K_{zz} = 3 - \langle\sigma\rangle^2 \quad (46)$$

$$I^H = \sum_{\substack{j,k=1 \\ j<k}}^{3}\left(K_{jj}K_{kk} - K_{jk}^{\,2}\right) = 3 - 2\langle\hat{\sigma}\rangle^2 \quad (47)$$

$$\det(K) = K_{xx}K_{yy}K_{zz} + 2K_{xy}K_{xz}K_{yz} - K_{xx}K_{yz}^{\,2} - K_{yy}K_{xz}^{\,2} - K_{zz}K_{xy}^{\,2} = \\ = 1 - \langle\hat{\sigma}\rangle^2 \quad (48)$$

with $\langle\hat{\sigma}\rangle^2 = \left\langle\hat{\sigma}_x\right\rangle^2 + \left\langle\hat{\sigma}_y\right\rangle^2 + \left\langle\hat{\sigma}_z\right\rangle^2$. As the correlation matrix is a positive definite one, these three invariants are all positive. In particular,the invariant (47) is the generalized uncertainty principle of ref. [7] (see Eq. (24)). Then, it is true that

$$I^H = 3 - 2\langle\hat{\sigma}\rangle^2 \geq 1 \quad (49)$$

which implies the well-known condition $\langle\hat{\sigma}\rangle^2 \leq 1$. As the state operator (39) describes a quantum mixed state, the semi-quantum system (31) can never be found in the pure state $\langle\hat{\sigma}\rangle^2 = 1$. Indeed, were it so, the positive definiteness of the correlation matrix would collapse $\det(K) = 1 - \langle\hat{\sigma}\rangle^2 \to 0$ as $\langle\hat{\sigma}\rangle \to 1$. We can then regard the invariant (48)) as an "indicator" that tells us if the system stays in a semi-quantum regime, without recourse to any other consideration. The invariance condition imposed on $\langle\hat{\sigma}\rangle^2$ by the uncertainty principle (49), together with the invariance of the energy $\langle\hat{H}\rangle$, strongly constrain the possible initial conditions of the classical variables. This is still another evidence of the interplay between both kind of variables. With the help of Eqs. (36), (37), and (38), it is easy to demonstrate that the quantity $|\vec{\alpha}| = \sqrt{\lambda_x^{\ 2} + \lambda_y^{\ 2} + \lambda_x^{\ 2}}$ is a constant of the motion. We will now establish the connection between invariant $|\vec{\alpha}|$ and invariant $\langle\hat{\sigma}\rangle^2$. Eqs. (33) to (38) indicate that

$$\langle\hat{\sigma}\rangle^2 = \langle\hat{\sigma}_x\rangle^2 + \langle\hat{\sigma}_y\rangle^2 + \langle\hat{\sigma}_z\rangle^2 = \tanh^2\left(\sqrt{\lambda_x^{\ 2} + \lambda_y^{\ 2} + \lambda_x^{\ 2}}\right) \quad (50)$$

which leads to

$$0 < \tanh^2\left(\sqrt{\lambda_x^{\ 2} + \lambda_y^{\ 2} + \lambda_x^{\ 2}}\right) < 1 \quad (51)$$

Eq. (51) is the uncertainty principle for the dual *Λ-space* of Lagrange multipliers whose expression is also affected by the classical degrees of freedom (see Eqs. (36) to (38)).

## 5. NUMERICAL SIMULATIONS

In order to illustrate the dynamics of the SU(2) Hamiltonians described in the previous Section we performed some numerical simulations. We have studied the set of non linear equations of motion corresponding to our Example (see Eqs. (33)-(35) and Eqs. (44)-(45)). One needs to set adequate initial conditions $\langle\hat{\sigma}_x\rangle_{(0)}$, $\langle\hat{\sigma}_y\rangle_{(0)}$, and $\langle\hat{\sigma}_z\rangle_{(0)}$ to illustrate the workings of the system as governed by the generalized uncertainty principle (GUP) of Eq. (30) and consider a large range of initial conditions (IC) as the polarization vector runs over the possible values allowed by the requirement $0 < \langle\hat{\sigma}\rangle^2 < 1$). The corresponding IC on the classical degrees of freedom were imposed by arbitrarily selecting the $q_{(0)}$-value while its partner $p_{(0)}$ was obtained from the energy conservation conditions (43). We have evaluated the Poincaré sections for the quantum state of the system (the Bloch sphere) for the situation in which *q = 0*. The results displayed in Figs. 1(1figura 1) and 2 (2figura 2 nueva) correspond to the following values: $\langle\hat{H}\rangle = 0.5$; *B* = 0.5; C = 1; *m* = 16; $q_{(0)} = 0.5$; $\langle\hat{\sigma}_x\rangle_{(0)} = \langle\hat{\sigma}_y\rangle_{(0)} = 0$; $\langle\hat{\sigma}_z\rangle_{(0)} = \langle\hat{\sigma}\ \rangle$,

$$p_{(0)} = \sqrt{2m\left(\langle\hat{H}\rangle - B\langle\hat{\sigma}_z\rangle_{(0)} - Cq_{(0)}\langle\hat{\sigma}_x\rangle_{(0)} - \frac{q_{(0)}^{\ \ 4}}{4}\right)}$$

was obtained from the energy conservation (see Eq. (43)).

FIGURA 1

FIGURA 2nueva

As the $\langle\hat{\sigma}\rangle$ value decreases from $\langle\hat{\sigma}\rangle \to 1$ to $\langle\hat{\sigma}\rangle \to 0$, the system evolves from an irregular regime to a regular one as can be appreciated from the corresponding Poincaré sections. In fact, the GUP imposes very strong constraints on the SU(2) non linear dynamics in the sense that, as the quantum degrees of freedom evolve, the polarization vector $\langle\hat{\sigma}\rangle$ must be confined to evolve on the surface of a Bloch of radius $\langle\hat{\sigma}\rangle^2 = \langle\hat{\sigma}_x\rangle^2 + \langle\hat{\sigma}_y\rangle^2 + \langle\hat{\sigma}_z\rangle^2 < 1$. The radius is constant and fixed at *t = 0* through the initial conditions.

## 6. CAN THE QUANTUM STATE OF THE SU(2) NON-LINEAR SEMI-QUANTUM SYSTEM BE CHAOTIC?

We have dealt here with time-independent semi quantum Hamiltonians. The quantum subsystem closes a partial Lie algebra under commutation with the Hamiltonian through Eq. (3). The MEP approach makes it clear that: i) when a Lie algebra can be associated to a semi quantum non-linear Hamiltonian through the closure condition, then it is possible to integrate the quantum degrees of freedom of this semi quantum non-linear system, ii) for the SU(2) Lie algebra case, this semi quantum non-linear dynamics exhibits some invariants which are very helpful to study the transition from regular to irregular dynamics. One of these invariants is the generalized uncertainty principle (GUP) of refs. [5,7], given that its constant value is fixed through the initial conditions. We ask whether it is possible to associate the irregular SU(2) non-linear dynamics of the quantum state (depicted in Figs. 1(1figura 1) and 2 (2figura 2 nueva)) to semi quantum chaos or we instead face a new kind of complex dynamics. The natural procedure would involve computing Lyapunov exponents. However, it was shown in ref. [31] that for this kind of Lie-governed dynamics this is not possible. Thus, in trying to provide a tentative answer, we have employed a wavelet statistical complexity analysis [32] (for details, see the Appendix I). This kind of analysis yields a probability distribution associated to the signals provided by our system's treatment, which in turn enables one to compute information quantifiers like entropy and statistical complexity. It is known [33] that chaotic systems are endowed with medium entropy values and large statistical complexities.

In Figs. 3(3SxComplejidad-Entropia), 4(4SyComplejidad-Entropia), and 5(5SzComplejidad-Entropia) we have depicted the Statistical Wavelet Complexity (*Complexity*) versus the relative entropy $S/S_{\max}$ for the time series associated with, respectively, $\langle\hat{\sigma}_x\rangle$, $\langle\hat{\sigma}_y\rangle$, and $\langle\hat{\sigma}_z\rangle$ for the following values of coupling constant C = 0.5; C = 1; C = 2 and C = 4. The value C = 1 corresponds to Poincaré surfaces of Figs. 1(1figura 1) and 2(2figura 2 nueva). We appreciate the fact that both the statistical complexity and the relative entropy are too low to be associated with chaos, according to the tenets of [33]. Thus, we tentatively conclude that our system does not display chaotic motion. The initial conditions used to generate Figs. 3(3SxComplejidad-Entropia),4 (4SyComplejidad-Entropia), and 5 (5SzComplejidad-Entropia) were $\langle\hat{\sigma}_x\rangle_{(0)} = \langle\hat{\sigma}_y\rangle_{(0)} = 0$; $\langle\hat{\sigma}_z\rangle_{(0)} = \langle\hat{\sigma}\rangle$ (i.e. the uncertainty principle invariant of Eq. (30)); $q_{(0)} = 0.5$ and $p_{(0)}$ as indicated above. The parameter values used were $\langle\hat{H}\rangle = 0.5$; B = 0.5; m = 16 and the coupling constant C was varied as it was said before.

FIGURA 3Sx Complejidad-Entropia
FIGURA 4Sy Complejidad_Entropia
FIGURA5 Sz Complejidad_Entropia

**7. CONCLUSIONS**

We can summarize our results as follows:

1.When it is possible to associate to the semi quantum non-linear system a Lie algebra under commutation operation with the Hamiltonian (through Eq. (3)), then it is always possible to find a statistical operator $\hat{\rho}(t)$ of maximum entropy for all times for the system, then it is possible to integrate the quantum degrees of freedom of the non-linear system. Particularly, if the semi quantum matrix $G(q\ ,p\ )$ (generated through the closure condition) is an anti-symmetric one, the non-linear semi quantum system possesses as many invariants of the motion as quantum degrees of freedom it has.

2.Eqs. (3), (9), (11), (20), and (21), produce an interplay between the quantum and classical degrees of freedom of the system. In this sense, we can say that the coupling constant *C* makes the quantum and classical variables of the system to be non separable.

3.The SU(2) dynamics of semi quantum Hamiltonians like Eqs. (2) or (28) exhibits interesting features that are unravelled with the help of our invariants of the motion, i.e.

it is always possible to find a set of non-commuting observable, through Eq. (3), to describe Hamiltonians of the type (28). This set is that of the generators of SU(2), i.e. $\{\langle\hat{\sigma}_x\rangle,\langle\hat{\sigma}_y\rangle,\langle\hat{\sigma}_z\rangle\}$. On account of them, the dynamics takes place in a semi quantum space of dimension 5: $V_{SQ} = gen\{\langle\hat{\sigma}_x\rangle,\langle\hat{\sigma}_y\rangle,\langle\hat{\sigma}_z\rangle,q,p\}$ and it is always possible to find the three invariants given by Eqs. (46), (47) and (48): the sum over the principal minors of order 1, 2, and 3 belonging to the correlation matrix *K(t)*. In particular, the sum over the principal minors of order 2 is the generalized uncertainty principle given by Eq. (24) [5].

These Hamiltonians exhibit the uncertainty principle as an invariant of the motion. This invariant takes the particular form $0 < \langle\hat{\sigma}\rangle^2 = \langle\hat{\sigma}_x\rangle^2 + \langle\hat{\sigma}_y\rangle^2 + \langle\hat{\sigma}_z\rangle^2 < 1$, whose value can be fixed by the initial conditions of the system.

From a wavelet statistical complexity analysis we tentatively conclude that our irregular regime is not chaotic.

4.The SU(2) dynamics displays two kind of regimes: a regular and an irregular one, as seen from the Poincaré sections of Figs. 1(1figura 1) and 2(2figura 2 nueva). The irregular regime always arises (for certain values of the system's parameters) when the invariant $\langle\hat{\sigma}\rangle^2 \to 1$. When the system attains an irregular regime, as the invariant runs from $\langle\hat{\sigma}\rangle^2 \to 1$ to $\langle\hat{\sigma}\rangle^2 \to 0$, the dynamics undergoes a regime-change from irregular to regular, according to the $\langle\hat{\sigma}\rangle^2$-value. As the condition $0 < \langle\hat{\sigma}\rangle^2 < 1$ is the uncertainty principle and $\langle\hat{\sigma}\rangle^2$ is a dynamic invariant of motion, we see that $\langle\hat{\sigma}\rangle^2$ is a regime-indicator, as Figs. 1(1figura 1) and 2(2figura 2 nueva) graphically explicate.

The initial conditions (IC′s) play a crucial role and deserve special attention. The IC′s should be chosen coherently, obeying the uncertainty relation given by Eq. (30) and the energy-conservation condition. The uncertainty principle also imposes constraints on the classical variables in the sense that the $p_{(0)}$-value cannot be determines without taking into account the condition (30).

5.Concerning what regime the system is in, as the Ehrenfest equations do not depend on $\hbar$, no limit: $\hbar \to 0$ is needed in our approach.

**APPENDIX 1: WAVELET STATISTICAL COMPLEXITY ANALYSIS**

The wavelet analysis (WA) is a powerful technique devised for the analysis of complex signals [32,34,35,36,37]. One introduces here the notions of statistical wavelet complexity and wavelet entropy. WA considers as a central concept that of a special function $\Psi(t)$ called the mother wavelet. From it one generates a wavelet family, $\Psi_{ab}(t)$. This is a set of elementary functions which arise out of dilations and translations of the mother wavelet $\Psi(t)$. An arbitrary element of the wavelet family can be expressed as [33,38]

$$\Psi_{ab}(t) = \frac{1}{\sqrt{a}}\Psi\left(\frac{t-b}{a}\right) \quad (52)$$

$a \neq 0$ and $b \in R$ are scale and translation parameters, respectively and $t$ is the time. As the scale parameter $a$ increases, the wavelet $\Psi_{ab}(t)$ becomes narrower. By modifying the $a$-value, it is possible to obtain several replicas, at different scales, with the variable time localization of a unique pattern [32,38]. As pointed out in ref. [38]: "for special choices of the mother wavelet function $\Psi(t)$, and for the discrete set of parameters $a_j = 2^{-j}$ and $b_{j,k} = 2^{-j/2}\,(j,k \in Z)$, the wavelet family $\Psi_{j,k}(t) = 2^{j/2}\Psi\left(2^j t - k\right)$ constitutes an orthonormal basis of Hilbert's space $L^2(\mathrm{R})$, consisting of finite-energy signals ". Our data points are those generated by the system of non-linear differential equations [Cf. Eqs. (33)-(35) and Eqs. (44)-(45)]. The wavelet analysis introduces a proper orthonormal basis so that any of our temporal signals *S(t)'s* (obtained through Eqs. (33)-(35) and Eqs. (44)-(45)) may be uniquely decomposed. Such decomposition may be carried out over all resolution levels *j* [32]. If we assume that the signal *S(t)* is given by the sampled values S={x(n); n = 1;...;N}, corresponding to a uniform grid with sampling time $T_S$ = 1, then the wavelet expansion, according to refs.[32], reads $\left(N_j = \log_2\left(N_j\right)\right)$

$$S(t) = \sum_{-N_j}^{-1}\left(\sum_k C_j(k)\Psi_{j,k}(t)\right) = \sum_{-N_j}^{-1} r_j(t) \quad (53)$$

where: $C_j(k)$ are the wavelet coefficients and the family $\Psi_{j,k}(t)$ is an orthonormal basis for $L^2(\mathrm{R})$, so that $C_j(k) = \left\langle S(t), \Psi_{j,k}(t)\right\rangle$. It is also possible to define the "energy" at each resolution level j = -1;...;-$N_j$ in the following fashion [32,36]

$$E_j = \left\|r_j(t)\right\|^2 = \sum_k \left|C_j(k)\right|^2 \quad (54)$$

and the total energy may be cast as

$$E_{TOT} = \left\|S(t)\right\|^2 = \sum_{j<0}\left(\sum_k \left|C_j(k)\right|^2\right) = \sum_{j<0} E_j$$

(55)

Eq. (54) measures the frequency-contribution corresponding to the resolution level *j*, while Eq. (55) measures the whole multiresolution level's contribution. Given that each energy level $E_j$ measures the contribution of the level j to the whole signal *S(t)*, it is possible to define a probability distribution $\{p_i\}_{-N_j \le i -1}$ for the multiresolution levels

$$p_j = \frac{E_j}{E_{TOT}} = \frac{\sum_k \left|C_j(k)\right|^2}{\sum_{j<0}\left(\sum_k \left|C_j(k)\right|^2\right)} = \frac{\sum_k \left|C_j(k)\right|^2}{\left\|S(t)\right\|^2}$$

(56)

$p_j$ is the probability that the signal *S(t)* main contain frequencies belonging to the multirresolution level *j* and $\{p_i\}_{-N_j \le i -1}$ is the probability distribution of energies (frequencies), that must obey the condition

$$\sum_{j=-N_j}^{-1} p_j = 1 \quad (57)$$

The total wavelet entropy (TWS) associated to the probability distribution is defined as [32]

$$S_{WT} = -\sum_{j<0} p_j \log_2\left(p_j\right) \quad (58)$$

As it was pointed out in ref. [32]: "the total wavelet entropy (58) appears as a measure of the degree of order/disorder of the signal. It provides useful information about the underlying dynamic process associated to the signal": one generated by a totally random or chaotic process can be taken as representative of a very disordered behavior: $S_{WT} \to S_{MAX} = \log_2\left(N_j\right)$. Conversely, an ordered process may be represented by a periodic mono-frequency signal: $S_{WT} \to 0$. In the wake of ref. [32], we will here adopt the definition of "disorder-amount" $Q$ (normalized total wavelet entropy, so that $0 \le Q \le 1$) as

$$Q = \frac{S_{WT}}{S_{MAX}} = \frac{S_{WT}}{\log_2\left(N_j\right)} = \frac{-\sum_{j<0} p_j \log_2\left(p_j\right)}{\log_2\left(N_j\right)} \quad (59)$$

and the definition of statistical wavelet complexity *Complexity* as [32]

$$Complexity = DQ; \qquad D = \sum_{j<0}\left(p_j - p_{eq}\right)^2 = \sum_{j<-N_j}^{-1}\left(p_j - \frac{1}{N}\right)^2 \quad (60)$$

where $D$ is the so-called "disequilibrium" of refs.[32,39], which measures "how far" the probability distribution $\{p_i\}$ is located from the uniform distribution $p_{eq} = \frac{1}{N_j}$ that characterizes equilibrium in Gibbs' statistical mechanics [32]. The statistical wavelet Complexity is also normalized: $0 \le Complexity \le 1$.

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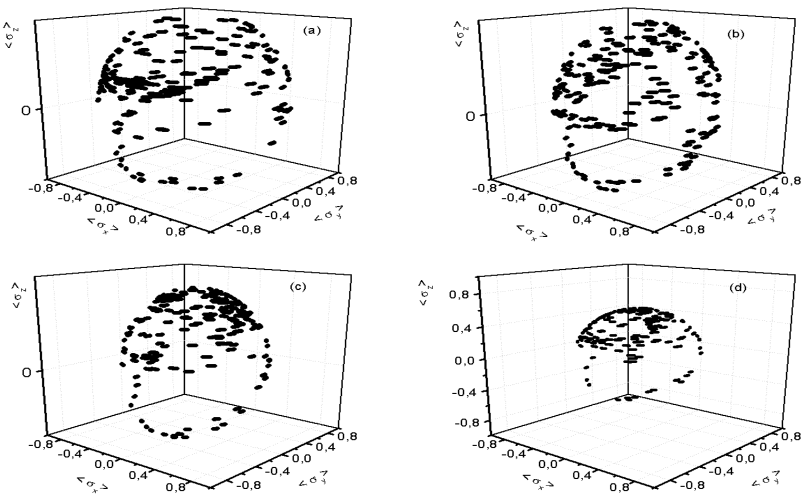


Fig. 1: Poincaré section on Bloch sphere for <H> = 0.5; B = 0.5; C = 1; m =16; $q_{(0)}$=0.5;$\langle\sigma_x\rangle_{(0)} = \langle\sigma_y\rangle_{(0)} = 0$ and $\langle\sigma_z\rangle_{(0)}$ : (a): 0.9; (b): 0.8; (c): 0.7; (d): 0.6

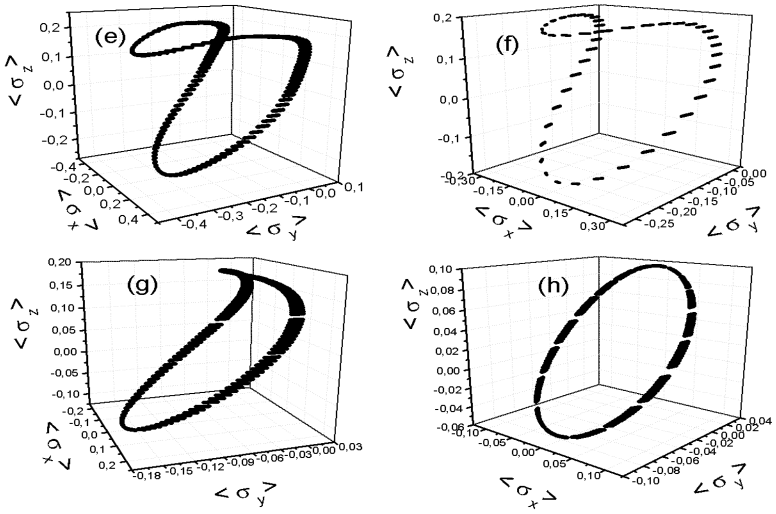


Fig. 2: Poincaré section on Bloch sphere for <H> = 0.5; *B* = 0.5; *C* = 1; *m* = 16; $q_{(0)} = 0.5$;$\langle\sigma_x\rangle_{(0)} = \langle\sigma_y\rangle_{(0)} = 0$ and $\langle\sigma_z\rangle_{(0)}$ : (e): 0.4; (f): 0.3; (g): 0.2; (h): 0.1

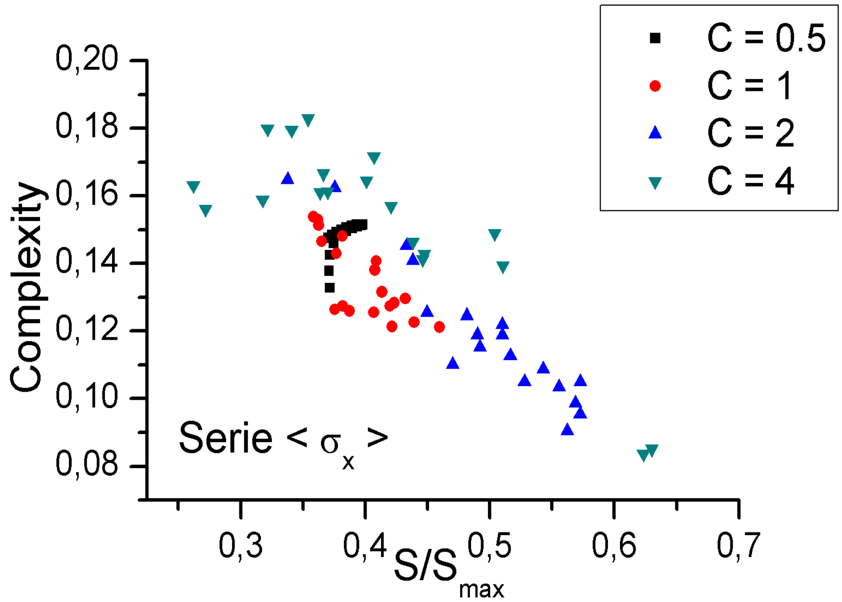


Fig.3: Statistical Wavelet Complexity vs. relative Entropy $S/S_{max}$ for $\langle\sigma_x\rangle$ time series for the following values of coupling constant: C = 0.5; C = 1; C = 2; C = 4

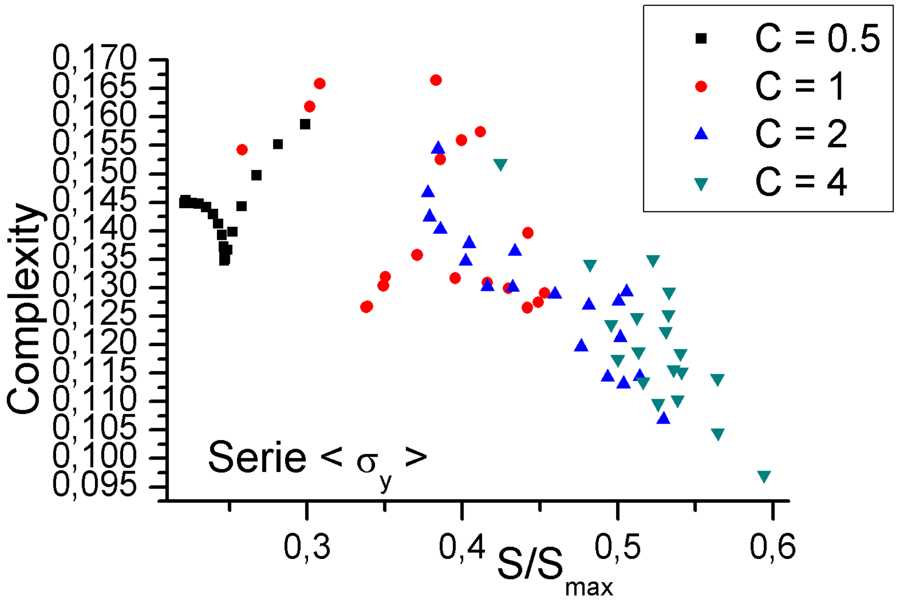


Fig.4: Statistical Wavelet Complexity vs. relative Entropy $S/S_{max}$ for $\langle\sigma_y\rangle$ time series for the following values of coupling constant: C = 0.5; C = 1; C = 2; C = 4

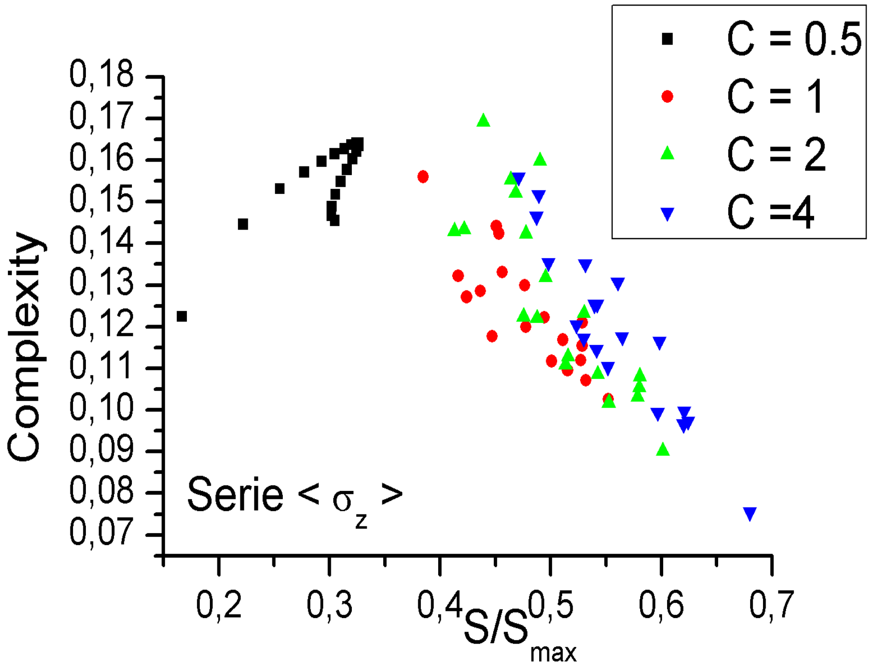


Fig.5: Statistical Wavelet Complexity vs. relative Entropy $S/S_{max}$ for $\langle\sigma_z\rangle$ time series for the following values of coupling constant: C = 0.5; C = 1; C = 2; C = 4