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The Derivation of Interval Type-2 Fuzzy Sets and Systems on Continuous Domain: Theory and Applications to Heart Diseases

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Abstract: An overview and a derivation of interval type-2 fussy logic system (IT2 FLS), which can handle rule's uncertainties on continuous domain, having good number of applications in real world. This work focused on the performance of an IT2 FLS that involves the operations of fuzzification, inference, and output processing. The output processing consists of Type-Reduction (TR) and defuzzification. This work made IT2 FLS much more accessible to FLS modelers, because it provides mathematical formulation for calculating the derivatives. Presenting extend to representation of T2 FSs on continuous domain and using it to derive formulas for operations, we developed and extended the derivation of the *union* of *two* IT2 FSs to the derivation of the *intersection* and *union* of *N*-IT2 FSs that is based on various concepts. The derivation of all the formulas that are related with an IT2 and these formulas depend on continuous domain with multiple rules. Each rule has multiple antecedents that are activated by a crisp number with T2 singleton fuzzification (SF). Then, we have shown how those results can be extended to T2 non-singleton fuzzification (NSF). We are derived the relationship between the consequent and the domain of uncertainty (DOU) of the T2 fired output FS. As well as, provide the derivation of the general form at continuous domain to calculate the different kinds of type-reduced. We have also applied an IT2 FLS to medical application of Heart Diseases (HDs) and an IT2 provide rather modest performance improvements over the T1 predictor. Finally, we made a comparison of HDs result between IT2 FLS using the IT2FLS in MATLAB and the IT2 FLS in Visual C# models with T1 FISs (Mamdani, and Takagi-Sugeno).

Keywords: Type-2 fuzzy sets, Interval type-2 membership functions, Footprint of uncertainties, set theoretic operations, Type-2 fuzzy logic system, Type-reduction, Heart diseases

1. INTRODUCTION

This work, introduced a new class of fuzzy logic systems—*interval type-2 fuzzy logic system* (IT2 FLS), where the antecedent or/and consequent membership functions (MFs) are interval type-2 fuzzy sets (IT2 FSs), [Mendel et al. 2009, 2006 and 2004], which is an extension of the concept of a *type-1 fuzzy set* (T1 FS). In an IT2 FLS, the knowledge used to construct rules is uncertain, and this uncertainty drives to rules having uncertain antecedents and/or consequents, [Wu and Mendel 2012, 2007 and 2002]. Now as MFs of a general T2 FSs are fuzzy, therefore T2 FSs are able to model as uncertainties, and their MFs are three-dimensional, [Zeng et al. 2008]. T2 FSs third dimension provides additional degrees that make it possible to directly models uncertainties, [Liang and Mendel 2000]. T2 FSs are difficult to use and understand because: i) T2 FSs three-dimensional makes them very difficult to depict; ii) there is no simple terms set that let us effective communication about T2 FSs, and to then be mathematically accurate, and iii) using T2 FSs is computationally more complex than using T1 FSs, [Mendel et al. 2009, 2007 and 2002]. Most people only use an IT2 FSs in a T2 FLS, because of the computational complex of using a general T2 FS, the result being an IT2 FLS. The resulting IT2 FLS have the chance to provide better performance than a T1 FLS, and all of the results that are needed to perform an IT2 FLS can be obtained by T1 FS mathematics. The computations related with IT2 FSs are very flexible, which makes an IT2 FLS to a large degree practical, [Melgarejo et al. 2004]. Section **2**, defined a small set of concepts in a mathematically accurate way of general T2 FSs and IT2 FSs. We are extended the theorem1, which was given by Mendel et al. 2006 for discrete universes of discourse, to continuous universes of discourse. Section **3,** derived the formulas of the *intersection* and *union* of *N*-IT2 FSs that is based on different concepts: i) the concept of embedded IT2 FSs such as theorem 2; ii) the concept of *Extension Principle* such as theorem 3. Additionally, we derived the formulas of the *meet* and *join* of *N*-IT2 FSs such as theorem 4, [Karnik et al. 2001, 1999, and 1998]. Section **4** has described an IT2 FLS, T2 singleton fuzzification (SF) and T2 non-singleton fuzzification (NSF). Present the derivation of all of the formulas that are related with an IT2 FLS at continuous domain, and handled multiple rules.

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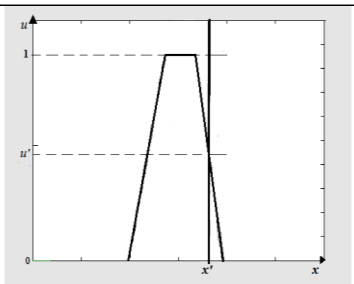
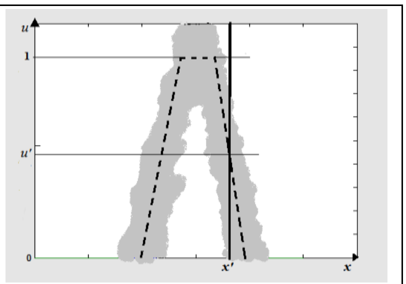
Each rule has multiple antecedents that are activated by a crisp number (the case of SF), after which we shown how those results can be extended to (the case of NSF), [Mendel et al. 2009, 2007, 2006 and 2004]. Consequently, we are derived the relationship between the consequent and the domain of uncertainty (DOU) of the T2 fired output FS that summarized by theorem 5a and 5b for SF and NSF, respectively, [Castro et al. 2008]. Section **5**, showed that computation of the continuous version of type-reduction that is used in going from fired-rule IT2 FSs to the defuzzified number at the final output of FLS, [Salazar et al 2011], [Karnik and Mendel 1999 and 2001]. We have provided the derivation of the general form for continuous domain to calculate the different kinds of type-reduced, which was given by Karnik et al. 2004 but for discrete domain. Additionally, we are presented the term of defuzzification which using the average of endpoints to obtain the crisp output of IT2 FLS, [Mendel et al. 2009, 2007 and 2002]. In Section **6**, a medical application of IT2 FLS's to heart diseases (HDs) is applied, which demonstrated the basic ideas and the mathematical operations of IT2 fuzzy sets and systems. We also provide a Matlab performance of IT2 FLS. A comparison of HDs between IT2 FLS using the IT2FLS in MATLAB and the IT2FLS in Visual C# models with T1 FISs (Mamdani, and Takagi-Sugeno) are presented in this Section. Section **7**, we draw conclusions. Finally, an Appendix is presents the concept of Extension Principle.

2. INTERVAL TYPE-2 FUZZY SETS

Most people only use interval type-2 fuzzy sets (IT2 FSs) in a type-2 fuzzy logic system (T2 FLS) because of the computational complexity of using a general T2 FS, the result being an interval type-2 fuzzy logic system (IT2 FLS). We define an IT2 FS and some important related concepts, to provide a simple collection of mathematically terms that will let us effectively communicate about such sets. Imagine fuzzing the type-1 membership function (MF) depicted through Fig. 1(a) by moving the points on the trapezoid either to the right or to the left with the different amounts, as in Fig. 1(b). Therefore, at a specific value of x, say x' for all $x \in X$, there no longer is a single value for the MF; instead, the MF takes on values wherever the vertical line intersects the fuzzy. The basic concepts of type-2 fuzzy sets are introduced as follows, [Mendel et al. 2009, 2007, and 2002]:



(a)



(b)

Fig. 1. (a) Representing the Type-1 MF and (b) Fizzing T1 MF

Definition 1: A type-2 fuzzy set, expresses the non-deterministic truth degree with uncertainty for an element that belongs to a set. A type-2 fuzzy set denoted by $\hat{A}$ is described by a *type-2 MF* $\mu_{\hat{A}}(x,u)$, where $x \in X$ and $u \in J_x^u \subseteq [0,1]$, and $0 \leq \mu_{\hat{A}}(x,u) \leq 1$ is defined as the following:

$$\hat{A} = \{((x,u), \mu_{\hat{A}}(x,u)) \mid \forall x \in X, \forall\ u \in J_x^u \subseteq [0,1]\} \tag{1}$$

It can also be expressed as:

$$\hat{A} = \int_{x\in X} \int_{u\in J_x^u} \mu_{\hat{A}}(x,u)/(x,u), \quad J_x^u \subseteq [0,1], \tag{2}$$

where $\iint$ denotes union over all allowable x and u. When uncertainties disappear, a T2 MF must reduce to a T1 MF, in which the variable u equals $\mu_{\hat{A}}(x,u)$, and the third dimension disappears. The amplitudes of a MF should locate between or be equal to zero and one. If all $\mu_{\hat{A}}(x,u)$ equal to one then $\hat{A}$ is an *interval T2 FS* (IT2 FS).

Definition 2: At each value of x, say $x = a$, the 2-dimentional plane whose axes are u and $\mu_{\bar{A}}(a,u)$ is called a vertical-slice of $\mu_{\hat{A}}(x,u)$. A secondary MF is a vertical-slice of $\mu_{\hat{A}}(x,u)$. It is $\mu_{\hat{A}}(x = a,u)$ for $a \in X$ and $\forall\ u \in J_x^u \subseteq [0,1]$, i.e,

$$\mu_{\hat{A}}(x = a,u) \equiv \mu_{\hat{A}}(a) = \int_{u\in J_x^u} 1/u\,, J_x^u \subseteq [0,1] \tag{3}$$

Because $\forall\ a \in X$, we drop the prime notation on $\mu_{\hat{A}}(a)$, and refer to $\mu_{\hat{A}}(x)$ as a secondary MF, the IT2 FS can be expressed (2) as, [Mendel et al. 2009, 2007, and 2002]:

$$\hat{A} = \int_{x\in X} \mu_{\hat{A}}(x)/x = \int_{x\in X} \left[\int_{u\in J_x^u} 1/u \right] \Big/ x. \tag{4}$$

The domain of a secondary MF (J_x^u) is called the primary membership of x, where $J_x^u \subseteq [0,1]$ for $\forall\, a \in X$.

Definition 3: The *amplitude* of a secondary MF is called a *secondary degree*. The secondary degrees of an IT2 FS are all equal to one. If x and J_x^u are both continuous, then the right-hand side of (4) can be denoted as:

$$\hat{A} = \int_{x\in X} \left[\int_{u\in J_x^u} 1/u \right] \Big/ x = \int_{i=1}^{N} \left[\int_{u\in J_x^u} 1/u \right] \Big/ x_i = \left[\int_{j=1}^{M_1} 1/u_{1j} \right] \Big/ x_1 \cup \ldots \cup \left[\int_{j=1}^{M_N} 1/u_{Nj} \right] \Big/ x_N, \tag{5}$$

where $\cup$ denotes the union, and N is an approach infinity. Note that, the variable x has been divided into N values, and at each of this value u has been divided into M_i values.

Definition 4: Uncertainty in the primary memberships of an IT2 FS $\hat{A}$ consists of a bounded region that we call the footprint of uncertainty (FOU). It is the union of all primary memberships as the following, Wu and Mendel 2012, 2007, and 2002:

$$FOU(\hat{A}) = \bigcup_{x\in X} J_x^u\ . \tag{6}$$

Equation (6) represents a *vertical-slice of the FOU*, because each of J_x^u is a vertical slice. The shaded region on the xu plane in Fig. 2 is the FOU. If a T2 FS is continuous with a naturally ordered primary variable then the *domain of uncertainty* (DOU) for a T2 FS equal to *FOU*, i.e., $DOU(\hat{A}) = FOU(\hat{A})$.

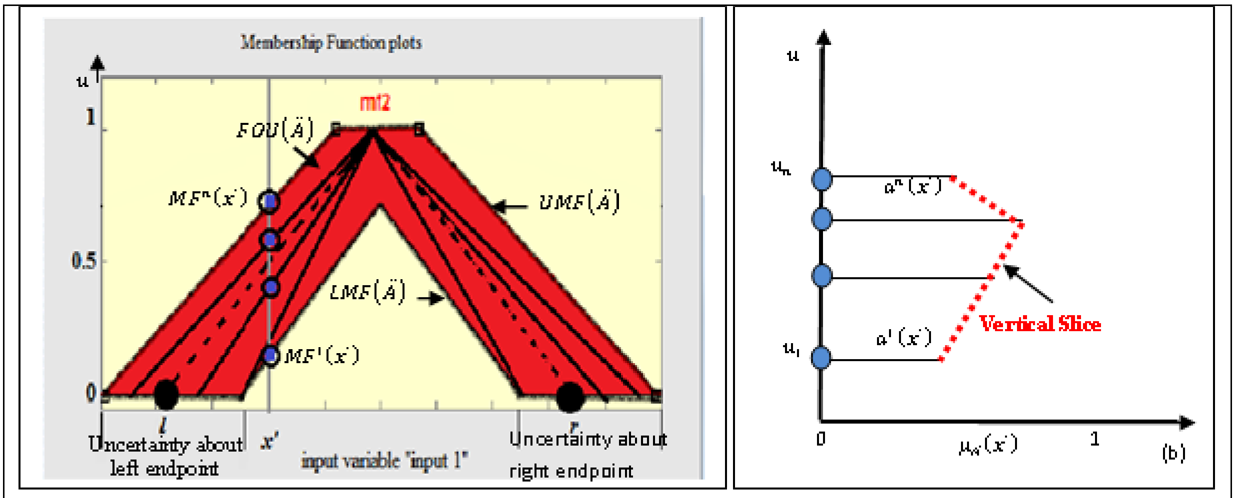


Fig. 2. (a) MFs when base endpoints *l* and *r* have uncertain intervals related with them, (b) The depicted of the vertical slice at *x'*.

Definition 5: The *lower membership function* (LMF) and *upper membership function* (UMF) of $\hat{A}$ are two T1 MFs that bound the DOU. The LMF is associated with the lower bound of $DOU(\hat{A})$ and is denoted by $\underline{\mu}_{\hat{A}}(x), \forall\, x \in X$ and the UMF is associated with the upper bound of $DOU(\hat{A})$ and is denoted by $\overline{\mu}_{\hat{A}}(x), \forall\, x \in X$, as following:

$$\underline{\mu}_{\hat{A}}(x) \equiv \underline{DOU(\hat{A})}, \text{ and } \overline{\mu}_{\hat{A}}(x) \equiv \overline{DOU(\hat{A})}, \qquad \forall\, x \in X \tag{7}$$

We observe that for an IT2 FS $J_x^u = \left[\underline{\mu}_{\hat{A}}(x),\ \overline{\mu}_{\hat{A}}(x) \right], \forall\, x \in X$. Thus, interval type-2 fuzzy set is denoted by:

$$\hat{A} = \int_{x\in X} \left[\int_{u\in\left[\underline{\mu}_{\hat{A}}(x),\overline{\mu}_{\hat{A}}(x)\right]\subseteq[0,1]} 1/u \right] \Big/ x \tag{8}$$

Definition 6: For continuous universes of discourse X and U, an embedded IT2 FS has $N \to \infty$ countable-infinity number of elements, where $\hat{A}_E$ contains exactly one element from $J_{x_i}^u$, namely u_i, $(i = 1,2, \ldots)$, each with a secondary degree equal to one, i.e.,

$$\hat{A}_E = \int_{i=1}^{\infty} [1/u_i]/x_i\,,\ u_i \in J_{x_i}^u \subseteq U = [0,1] \tag{9}$$

Set $\hat{A}_E$ is an embedded in $\hat{A}$. There are a countable-infinite number of embedded IT2 FSs for a continuous IT2.

Theorem 2.1: Let $\hat{A}_E^k$ denote the k^{th} embedded IT2 FSs for T2 FS, when X and U are continuous, is as follows:

$$\hat{A}_E^k \equiv \left\{\left(u_i^k, \mu_{\hat{A}}(x_i, u_i^k)\right), i = 1, \dots, \infty\right\}, \quad (10)$$

where $u_i^k \in \{u_{ij}, j = 1, \dots, M_i\}$, then $\hat{A}$ is the union of all of its embedded IT2 FSs, i.e,

$$\hat{A} = \int_{k=1}^{n_{N\to\infty}} \hat{A}_E^k \quad (11)$$

in which $n_{N\to\infty} \equiv \prod_{i=1}^{\infty} M_i$, and M_i denotes the partition levels of secondary variable u_i^k at each of the x_i.

Proof:

We prove this theorem by proving that the general case of Equation (6) can be re-expressed so that it contains exactly all of the parts on the right-hand side of (11). Firstly, we should be proving for each node along the u axis is contained in $n_l = \prod_{\substack{i=1 \\ i\neq l}}^{\infty} M_i$ embedded sets $(l = 1, \dots \infty)$. Because all embedded sets start with an element along the u_1-axis, and each element along that axis spread out into $\prod_{i=2}^{\infty} M_i$ embedded sets, then we observe that

$$\prod_{i=2}^{\infty} M_i = \prod_{\substack{i=1 \\ i\neq 1}}^{\infty} M_i \equiv n_1, \quad (12)$$

Therefore, ponder elements along the u_2 axis. The M_1elements along the u_1 axis spread out into all the elements along the u_2 axis after which each element along the u_2 axis spread out into $\prod_{i=3}^{\infty} M_i$ embedded sets. Therefore, there are a total of $M_1 \prod_{i=3}^{\infty} M_i$ embedded sets for each node along the u_2 axis. Then, we obtain,

$$M_1 \prod_{i=3}^{\infty} M_i = \prod_{\substack{i=1 \\ i\neq 2}}^{\infty} M_i \equiv n_2, \quad (13)$$

Continuing in this way up to the u_N axis, where N is nearing to infinity. In this case, ponder elements are along the u_N axis. The M_{N-1}elements along the u_{N-1} axis spread out into all the elements along the u_N axis after which each element along the u_N axis spread out into $\prod_{i=N+1}^{\infty} M_i = \prod_{i=\infty+1}^{\infty} M_i$ embedded sets. This means that there are a total of $\prod_{i-1}^{N-1} M_i * \prod_{i=N+1}^{\infty} M_i$ embedded sets for each node along the $u_{N\to\infty}$ axis.

Note that,

$$\prod_{i-1}^{N-1} M_i * \prod_{\substack{i=N+1 \\ i\neq N}}^{\infty} M_i \equiv n_{N\to\infty}, \quad (14)$$

so, we proved

$$n_l = \prod_{\substack{i=1 \\ i\neq l}}^{\infty} M_i \quad (15)$$

Next, what we do is to repeat term 1 in (5) n_1 times, upto last term in (5) $n_{N\to\infty}$ times (since $\hat{A}_i \cup \hat{A}_i = \hat{A}_i$), as the following:

$$\hat{A} = \int_{i=1}^{n_1} \left\{\left[\int_{j=1}^{M_1} 1/u_{1,j}\right]\Big/ x_1\right\} \cup \dots \cup \int_{i=1}^{n_{N\to\infty}} \left[\int_{j=1}^{M_{N\to\infty}} 1/u_{N\to\infty,j}\right]\Big/ x_{N\to\infty} \quad (16)$$

Now, we must prove that (16) can be reorganized as in (11). We do this by proving that (16) has exactly the same countable-infinity number of elements as does (11). Since, each $\hat{A}_E^k$ has countable-infinity number of elements, and then $\hat{A}$ in (11) has $n_{N\to\infty} = (N\to\infty) * \prod_{i=1}^{N\to\infty} M_i$ elements. From (16), $\hat{A}$ has $n_1 M_1 \cup n_2 M_2 \cup \dots \cup n_{N\to\infty} M_{N\to\infty}$ of elements. However, from (15), we obtain:

$$n_1 M_1 \cup n_2 M_2 \cup \dots \cup n_{N\to\infty} M_{N\to\infty} = \prod_{i=1}^{N\to\infty} M_i \cup \prod_{i=1}^{N\to\infty} M_i \cup \dots \cup \prod_{i=1}^{N\to\infty} M_i = (N\to\infty) * \prod_{i=1}^{N\to\infty} M_i \quad (17)$$

Consequently, we have proved Equation (16) has the same countable-infinite number of elements such as (11).

Corollary 1:

We can express *T2 FS* for all of the secondary degrees of an IT2 FS as:

$$\hat{A} = 1/DOU(\hat{A}), \quad (18)$$

where

$$DOU(\hat{A}) = \int_{k}^{n_{N\to\infty}} A_E^k = \int_{k}^{n_{N\to\infty}} \int_{k}^{N\to\infty} u_i^k/x_i = \left\{\left[\underline{\mu}_{\hat{A}}(x), \overline{\mu}_{\hat{A}}(x)\right], \ \forall x \in X_c\right\}, \quad (19)$$

in which X_c is a continuous universe of discourse, which means the $DOU(\hat{A})$ contain a countable-infinity number of functions that completely fills the space between $\underline{\mu}_{\hat{A}}(x)$ and $\overline{\mu}_{\hat{A}}(x)$ for $\forall x \in X_c$.

Proof:

Since, $\hat{A}_E = \int_{i=1}^{N\to\infty} [1/u_i]/x_i$ and $A_E = \int_{i=1}^{N\to\infty} u_i/x_i$, then we obtain:

$$\hat{A}_E = 1/A_E \quad (20)$$

From (20), each A_E^k in (11) can be expressed as $1/A_E^k$, then,

$$\hat{A} = \int_{k}^{n_{N\to\infty}} 1/A_E^k = 1\Big/\int_{k}^{n_{N\to\infty}} A_E^k \equiv 1/DOU(\hat{A}) \quad (21)$$

3. SET-THEORETIC OPERATIONS ON TYPE-2 FUZZY SETS

The main aim of this section is to derive formulas for the *intersection* and *union* of N IT2 FSs of an IT2 FS, because these operations are used in an IT2 FLS. In this Section, the derivation of the *intersection* and *union* of N IT2 FSs is based on tow concepts: 1) the concept of embedded IT2 FSs such as theorem 2.1; 2) the concept of *Extension Principle* such as theorem 3. Third part contains the derivation of the *meet* and *join* of N IT2 FSs, [Karnik et al. 2001, 1999, and 1998], [Mendel et al. 2009, 2007, and 2002].

Theorem 3.1: Derivation the intersection of N-T2 FSs depending on the concept of the embedded IT2 FSs

The *intersection* and *union* of N IT2 FSs, $\hat{A}_i$ $(i = 1, \dots, N)$, are given by (22) and (23), respectively:

$$\bigcap_{i=1}^{N} \hat{A}_i = 1\Big/\left[\bigwedge_{i=1}^{N} \underline{\mu}_{\hat{A}_i}(x), \bigwedge_{i=1}^{N} \overline{\mu}_{\hat{A}_i}(x)\right], \forall\, x \in X. \quad (22)$$

$$\bigcup_{i=1}^{N} \hat{A}_i = 1\Big/\left[\bigvee_{i=1}^{N} \underline{\mu}_{\hat{A}_i}(x), \bigvee_{i=1}^{N} \overline{\mu}_{\hat{A}_i}(x)\right], \forall\, x \in X. \quad (23)$$

Proof:

Since the proofs of parts 1 and 2 are similar, therefore we only provide the proof for parts 1. Consider N IT2 FSs $\hat{A}_i$ $(i = 1, \dots, N)$. From Theorem1 and Corollary1, it is follow that:

$$\bigcap_{i=1}^{N} \hat{A}_i = \int_{k1=1}^{n_{A_1}} A_{1E}^{k1} \cap \dots \cap \int_{kN=1}^{n_{A_N}} A_{NE}^{kN} = \int_{k1=1}^{n_{A_1}} \dots \int_{kN=1}^{n_{A_N}} A_{1E}^{k1} \cap \dots \cap A_{NE}^{kN} = 1\Big/\mathrm{DOU}\left(\bigcap_{i=1}^{N} \hat{A}_i\right), \quad (24)$$

where n_{A_i} , $i = 1, \dots, N$ and $ki = 1, \dots, n_{A_i}$, denote the countable-infinity number of embedded IT2 FSs that are associated with $\hat{A}_i$, and

$$\mathrm{DOU}\left(\bigcap_{i=1}^{N} \hat{A}_i\right) = \int_{k1=1}^{n_{A_1}} \dots \int_{kN=1}^{n_{A_N}} A_{1E}^{k1} \cap \dots \cap A_{NE}^{kN} \quad (25)$$

Now, we must compute the intersection of the $n_{A_1} \times \dots \times n_{A_N}$ pairs of embedded T1 FSs A_{iE}^{ki}, $(i = 1, \dots, N)$. Recollect that the intersection of N T1 FSs is a function as the fowlloing:

$$\bigcap_{i=1}^{N} A_{iE}^{ki} = min\left\{\mu_{A_{1E}^{k1}}(x_j), \mu_{A_{2E}^{k2}}(x_j), \dots, \mu_{A_{NE}^{kN}}(x_j)\right\},\ \ j = 1, \dots, N. \quad (26)$$

Equation (25) is a set of $n_{A_1} \times \dots \times n_{A_N}$ functions that contain lower and upper bounding functions since all $\mu_{\hat{A}_{iE}^{ki}}(x_j)$ are bounded for all values of x_j. Each primary membership is defined over a continuous domain, $n_{A_i} \to \infty$, and the multiple infinite intersection of embedded T1 FSs in (25) contains lower and upper bounding functions, because $\hat{A}_i$ each have a bounded DOU. We now obtain formulas for these bounding functions. For each $\hat{A}_i$, $\underline{\mu}_{\hat{A}_i}(x)$ and $\overline{\mu}_{\hat{A}_i}(x)$ denote its lower MF and upper MF. Consequently, be true that

$$\inf_{\forall\, ki} min\left\{\mu_{A_{1E}^{k1}}(x_j), \mu_{A_{2E}^{k2}}(x_j), \dots, \mu_{A_{NE}^{kN}}(x_j)\right\} = min\left\{\underline{\mu}_{\hat{A}_1}(x), \underline{\mu}_{\hat{A}_2}(x), \dots, \underline{\mu}_{\hat{A}_N}(x)\right\},\quad \forall\, x \in X_c$$
$$= \bigwedge_{i=1}^{N} \underline{\mu}_{\hat{A}_i}(x),\ \ \forall\, x \in X_c \quad (27)$$

$$\sup_{\forall\, ki} min\left\{\mu_{A_{1E}^{k1}}(x_j), \mu_{A_{2E}^{k2}}(x_j), \dots, \mu_{A_{NE}^{kN}}(x_j)\right\} = min\left\{\overline{\mu}_{\hat{A}_1}(x), \overline{\mu}_{\hat{A}_2}(x), \dots, \overline{\mu}_{\hat{A}_N}(x)\right\},\quad \forall\, x \in X_c$$
$$= \bigwedge_{i=1}^{N} \overline{\mu}_{\hat{A}_i}(x),\ \ \forall\, x \in X_c \quad (28)$$

From (24)-(28), we obtained:

$$\bigcap_{i=1}^{N} \hat{A}_i = 1 \Big/ \int_{k1=1}^{n_{A_1}} \dots \int_{kN=1}^{n_{A_N}} A_{1E}^{k1} \cap \dots \cap A_{N_E}^{kN}$$

$$= 1\Big/\left[\underline{\mu}_{\hat{A}_1}(x) \wedge \dots \wedge \underline{\mu}_{\hat{A}_N}(x), \overline{\mu}_{\hat{A}_1}(x) \wedge \dots \wedge \overline{\mu}_{\hat{A}_N}(x)\right] = 1\Big/\left[\bigwedge_{i=1}^{N} \underline{\mu}_{\hat{A}_i}(x), \bigwedge_{i=1}^{N} \overline{\mu}_{\hat{A}_i}(x)\right] \quad (29)$$

Theorem 3.2: Derivation the intersection of N-T2 FSs depending on the concept of the Extension Principle Let $\hat{A}_i$ T2 FSs in a continuous universe X_c. Suppose $\mu_{\hat{A}_i}(x) = \int_u^{\cdot} g_{ix}(u_i)/u_i$, $(i = 1, \dots, N)$ and $\forall\, x \in X_c$ be the membership degrees of $\hat{A}_i$, where $u_i \in J_x^u$. Then membership degrees for intersection and union of type-2 FS have been defined as follows:

$$\bigcap_{i=1}^{N} \hat{A}_i \Leftrightarrow \mu_{\hat{A}_1 \cap \dots \cap \hat{A}_N}(x) = \int_{u_1 \in J_x^u} \dots \int_{u_N \in J_x^u} \left(g_{1x}(u_1) \star \dots \star g_{Nx}(u_N)\right) / T(u_1 \star \dots \star u_N) \quad (30a)$$

$$\bigcup_{i=1}^{N} \hat{A}_i \Leftrightarrow \mu_{\hat{A}_1 \cup \dots \cup \hat{A}_N}(x) = \int_{u_1 \in J_x^u} \dots \int_{u_N \in J_x^u} \left(g_{1x}(u_1) \star \dots \star g_{Nx}(u_N)\right) / T(u_1 \vee \dots \vee u_N) \quad (30b)$$

Proof:

We only provide the proof of (30a), because the proof of (30a) is similar to (30b). Consider N T2 FSs, $\hat{A}_i$ $(i = 1, \dots, N)$ in continuous universe of discourse. From the general case of (4) and since $\mu_{\hat{A}}(x,u) = \int_{u \in X} \mu_{\hat{A}}(x)/x$, we obtain:

$$\bigcap_{i=1}^{N} \hat{A}_i \Leftrightarrow \mu_{\hat{A}_1 \cap \dots \cap \hat{A}_N}(x,u) = \int_{x \in X} \mu_{\hat{A}_1 \cap \dots \cap \hat{A}_N}(x)/x$$

$$= \int_{x \in X} \left[\int_{u \in J_x^u \subseteq [0,1]} g_x(u)/u \right] \Big/ x \quad (31)$$

in which,

$$\int_{u \in J_x^u \subseteq [0,1]} g_x(u)/u$$

$$= T\left(\int_{u_1 \in J_x^u \subseteq [0,1]} g_{1x}(u_1)/u_1, \int_{u_2 \in J_x^u \subseteq [0,1]} g_{2x}(u_2)/u_2, \dots, \int_{u_N \in J_x^u \subseteq [0,1]} g_{Nx}(u_N)/u_N \right), \quad (32)$$

$$= \mathrm{T}\left(\mu_{\hat{A}_1}(x), \mu_{\hat{A}_2}(x), \dots, \mu_{\hat{A}_N}(x)\right),$$

where T is a t-norm function of the secondary membership functions because the intersection of N type-1 fuzzy sets is equivalent to the t-norm (e.g., minimum or product), i.e. $T_{i=1}^{N} \mu_{\hat{A}_i}(x)$, and $\mu_{\hat{A}_i}(x)$ are type-1 fuzzy sets. Observe that, from the *Extension Principle*, (see appendix a), equation (a-2) with (32), we obtain:

$$T\left(\int_{u_1 \in J_x^u}^{\cdot} g_{1x}(u_1)/u_1, \dots, \int_{u_N \in J_x^u}^{\cdot} g_{Nx}(u_N)/u_N \right)$$

$$= \int_{u_1 \in J_x^u}^{\cdot} \dots \int_{u_N \in J_x^u}^{\cdot} \left(g_{1x}(u_1) \star \dots \star g_{Nx}(u_N)\right) / T(u_1, \dots, u_N) \quad (33)$$

If T is the minimum operation $\star$, then $T(u_1, \dots, u_N) = u_1 \star \dots \star u_N$, therefore, when (33) is replaced into (31) for $\mu_{\hat{A}_1 \cap \dots \cap \hat{A}_N}(x)$ we obtain:

$$\mu_{\hat{A}_1 \cap \dots \cap \hat{A}_N}(x) = \int_{u \in J_x^u \subseteq [0,1]}^{\cdot} g_x(u)/u$$

$$\therefore \quad \bigcap_{i=1}^{N} \hat{A}_i \Leftrightarrow \mu_{\hat{A}_1 \cap \dots \cap \hat{A}_N}(x) = \int_{u_1 \in J_x^u} \dots \int_{u_N \in J_x^u} \left(g_{1x}(u_1) \star \dots \star g_{Nx}(u_N)\right) / T(u_1 \star \dots \star u_N) \quad (34)$$

$$\equiv \mu_{\hat{A}_1}(x) \sqcap \mu_{\hat{A}_2}(x) \sqcap \dots \sqcap \mu_{\hat{A}_N}(x) \qquad \forall\, x \in X_c \quad (35)$$

in which $\sqcap$ denotes the so-called *meet* operation. The use of the observation $\mu_{\hat{A}_1}(x) \sqcap \dots \sqcap \mu_{\hat{A}_N}(x)$ to indicate the meet between the secondary membership functions $\mu_{\hat{A}_i}(x)$. Therefore, we should be proving the *meet* operation between $\mu_{\hat{A}_i}(x)$ $(i = 1, \dots, N)$, in (35) using theorem 4, [Mendel et al. 2009, 2004 and 2002].

Theorem 3.3: Derivation the meet operations of N-T2 FSs depending on the concept of the secondary MF Suppose that we have n convex, normal, type-1 real fuzzy sets $\hat{A}_i$ described by membership functions a_i respectively. Let x_i be real numbers such that $x_1 \le x_2 \le \dots \le x_n$ and $a_1(x_1) \le \dots \le a_n(x_n)$, then,

$$\mu_{A_1 \sqcap \dots \sqcap A_n}(\alpha) = \bigcap_{i=1}^{N} \hat{A}_i = \begin{cases} \bigvee_{i=1}^{n} a_i(\alpha), & \alpha < x_1 \\ \bigwedge_{i=1}^{k} a_i(\alpha), & x_k \le \alpha < x_{k+1}, 1 \le k < n-1 \\ \bigwedge_{i=1}^{n} a_i(\alpha), & \alpha \ge x_n \end{cases} \quad (36)$$

Proof:
In general, evaluating the meet operation is difficult to do for arbitrary T2 FSs. When $n = 2$, the meet operation between A_1 and A_2 can be expressed as:

$$\bigcap_{i=1}^{2} A_i = \int_{x_1} \int_{x_2} \bigwedge_{i=1}^{2} a_i(x_i) \bigg/ \bigwedge_{i=1}^{2} x_i \quad (37)$$

For each pair of points $\{x_1, x_2\}$ such that $x_1 \in A_1$ and $x_2 \in A_2$, we compute the minimum of x_1 and x_2, and the minimum of their memberships, so that $x_1 \wedge x_2$ is an element of $A_1 \cap A_2$ and $a_1(x_1) \wedge a_2(x_2)$ is the corresponding degree of membership.
Every resulting set element is obtained as the min operation result on one or more $\{x_1, x_2\}$ pairs, and its membership is the minimum of all the min operation results on memberships of x_1 and x_1. When $\alpha \in A_1 \cap A_2$ is the minimum operation result on some pair $\{x_1, x_2\}$, s.t. $x_1 \in A_1$ and $x_2 \in A_2$, then the pairs being $\{x_1, \alpha\}$ where $x_1 \in [\alpha, \infty)$ and $\{\alpha, x_2\}$ where $x_2 \in [\alpha, \infty)$. The process of computing the membership of α in $A_1 \cap A_2$ can be divided into three steps:
(1) For all $x_1 \in [\alpha, \infty)$, find the minimum between the memberships, thus find their supermom.
(2) For all $x_2 \in [\alpha, \infty)$, find the minimum between the memberships, thus find their supermom.
(3) find the maximum of the supermoms that resulted from (1) and (2), as the following:

$$\mu_{A_1 \sqcap A_2}(\alpha) = \left(\sup_{x_1 \in [\alpha, \infty)} \{a_1(x_1) \wedge a_2(\alpha)\} \right) \vee \left(\sup_{x_2 \in [\alpha, \infty)} \{a_1(\alpha) \wedge a_2(x_2)\} \right) \quad (38)$$

$$= \left(a_2(\alpha) \wedge \sup_{x_1 \in [\alpha, \infty)} a_1(x_1) \right) \vee \left(a_1(\alpha) \wedge \sup_{x_2 \in [\alpha, \infty)} a_2(x_2) \right) \quad (39)$$

Next, we divide α into three intervals: $\alpha < x_1, \alpha \in [x_1, x_2)$, and $\alpha \ge x_2$

a. When $\alpha = \alpha_1 < x_1$, because a_1 and a_2 both are non-increasing in $[x_1,, \infty)$, $\sup_{x_1 \in [\alpha, \infty)} a_1(x_1) = a_1(\alpha)$ and $\sup_{x_2 \in [\alpha, \infty)} a_2(x_2) = a_2(\alpha)$; then, we obtain:
$$\mu_{A_1 \sqcap A_2}(\alpha) = a_1(\alpha) \vee a_2(\alpha), \quad \forall\, \alpha < x_1 \quad (40)$$
b. When $x_1 \le \alpha = \alpha_2 < x_2$, we remember that $a_1(x_1) = 1$ and a_2 is non-increasing in $[x_2,, \infty)$, thus, $\sup_{x_1 \in [\alpha, \infty)} a_1(x_1) = 1$ and $\sup_{x_2 \in [\alpha, \infty)} a_2(x_2) = a_2(\alpha)$, then, we obtain:
$$\mu_{A_1 \sqcap A_2}(\alpha) = a_2(\alpha) \vee [a_1(\alpha) \wedge a_2(\alpha)] = a_1(\alpha), \quad \forall\, x_1 \le \alpha < x_2 \quad (41)$$
c. When $\alpha = \alpha_3 \ge x_2$, a_1 and a_1 have achieved their minimum values; then, $\sup_{x_1 \in [\alpha, \infty)} a_1(x_1) = a_1(\alpha)$ and $\sup_{x_2 \in [\alpha, \infty)} a_2(x_2) = a_1(\alpha)$, therefore, we obtain:
$$\mu_{A_1 \sqcap A_2}(\alpha) = (a_2(\alpha) \wedge a_1(\alpha)) \vee (a_1(\alpha) \wedge a_2(\alpha)) = a_1(\alpha) \wedge a_2(\alpha), \quad \forall\, \alpha \ge x_2 \quad (42)$$

From (40)-(42), we obtain:

$$\mu_{A_1 \sqcap A_2}(\alpha) = \begin{cases} a_1(\alpha) \vee a_2(\alpha), & \alpha < x_1 \\ a_1(\alpha), & x_1 \le \alpha < x_2 \\ a_1(\alpha) \wedge a_2(\alpha), & \alpha \ge x_2 \end{cases} \quad (43)$$

Note that, because a_1 and a_2 are normal and convex with $a_1(x_1) = a_2(x_2) = 1$, therefore $a_1 \vee a_2$ is non-decreasing in $(-\infty, x_1]$, a_1 is non-increasing in $[x_1, x_2]$, $a_1 \wedge a_2$ is non-increasing in $[x_2, \infty)$.

From these three notations, we see that $\mu_{A_1 \sqcap A_2}$ is non-decreasing in $(-\infty, x_1]$ and non-increasing in $[x_1, \infty)$, then $\mu_{A_1 \sqcap A_2}$ is convex. As well as, $\mu_{A_1 \sqcap A_2}$ is normal with $\mu_{A_1 \sqcap A_2}(x_1) = 1$.
Next, when $n > 2$, from associative law, we obtain $A_1 \sqcap A_2 \sqcap A_3 = (A_1 \sqcap A_2) \sqcap A_3$, therefore, from (43) we have:

$$\mu_{A_1 \sqcap A_2 \sqcap A_3}(\alpha) = \begin{cases} \mu_{A_1 \sqcap A_2}(\alpha) \vee a_3(\alpha), & \alpha < x_1 \\ a_1(\alpha), & x_1 \le \alpha < x_2 \\ \mu_{A_1 \sqcap A_2}(\alpha) \wedge a_3(\alpha), & \alpha \ge x_3 \end{cases} \quad (44)$$

Since $x_2 \le x_3$, we can rewrite (43) and (44) as:

$$\mu_{A_1 \sqcap A_2}(\alpha) = \begin{cases} a_1(\alpha) \vee a_2(\alpha), & \alpha < x_1 \\ a_1(\alpha) & x_1 \le \alpha < x_2 \\ a_1(\alpha) \wedge a_2(\alpha), & x_2 \le \alpha < x_3 \\ a_1(\alpha) \wedge a_2(\alpha), & \alpha \ge x_3 \end{cases} \text{ and } \mu_{A_1 \sqcap A_2 \sqcap A_3}(\alpha) = \begin{cases} \mu_{A_1 \sqcap A_2}(\alpha) \vee a_3(\alpha), & \alpha < x_1 \\ a_1(\alpha) & x_1 \le \alpha < x_2 \\ \mu_{A_1 \sqcap A_2}(\alpha) \wedge a_3(\alpha), & x_2 \le \alpha < x_3 \\ \mu_{A_1 \sqcap A_2}(\alpha) \wedge a_3(\alpha), & \alpha \ge x_3 \end{cases}$$

$$\mu_{A_1 \sqcap A_2 \sqcap A_3}(\alpha) = \begin{cases} a_1(\alpha) \vee a_2(\alpha) \vee a_3(\alpha), & \alpha < x_1 \\ a_1(\alpha) & x_1 \le \alpha < x_2 \\ a_1(\alpha) \wedge a_2(\alpha), & x_2 \le \alpha < x_3 \\ a_1(\alpha) \wedge a_2(\alpha) \wedge a_3(\alpha), & \alpha \ge x_3 \end{cases} = \begin{cases} \bigvee_{i=1}^{3} a_i(\alpha), & \alpha < x_1 \\ \bigwedge_{i=1}^{k} a_i(\alpha), & x_k \le \alpha < x_{k+1}, 1 \le k < 2 \\ \bigwedge_{i=1}^{3} a_i(\alpha), & \alpha \ge x_3 \end{cases}$$

It is clear and direct to show that $A_1 \sqcap A_2 \sqcap A_3$ is also a convex and normal set, because it is depend $\mu_{A_1 \sqcap A_2}$ that was convex and normal set. Then, (43) can be applied again to obtain $\mu_{A_1 \sqcap A_2 \sqcap A_3 \sqcap A_4}$ as the following:

$$\mu_{A_1 \sqcap A_2 \sqcap A_3 \sqcap A_4}(\alpha) = \begin{cases} a_1(\alpha) \vee a_2(\alpha) \vee a_3(\alpha) \wedge a_4(\alpha), & \alpha < x_1 \\ a_1(\alpha) & x_1 \le \alpha < x_2 \\ a_1(\alpha) \wedge a_2(\alpha), & x_2 \le \alpha < x_4 \\ a_1(\alpha) \wedge a_2(\alpha) \wedge a_3(\alpha), & x_3 \le \alpha < x_4 \\ a_1(\alpha) \wedge a_2(\alpha) \wedge a_3(\alpha) \wedge a_4(\alpha), & \alpha \ge x_4 \end{cases}$$

$$= \begin{cases} \bigvee_{i=1}^{4} a_i(\alpha), & \alpha < x_1 \\ \bigwedge_{i=1}^{k} a_i(\alpha), & x_k \le \alpha < x_{k+1}, 1 \le k < 3 \\ \bigwedge_{i=1}^{4} a_i(\alpha), & \alpha \ge x_4 \end{cases} \quad (45)$$

Continuing in this modality, we obtain (36). From this approach note, it is based on modeling a secondary MF.

4. INTERVAL TYPE-2 FUZZY LOGIC SYSTEM

We assume that all the antecedent and consequent fuzzy sets in the rules are T2. A FLS is T2 as long as any one of its antecedent or consequent FSs is T2. The rules structure remains the same in the case of T2, but some or all of the FSs involved are T2, [Melgarejo et al. 2004], [Zeng et al. 2008]. The T2 FLS has n inputs $x_1 \in X_1, \ldots, x_n \in X_n$, and output $y \in Y$, and, is described by L rules, where the l^{th} rule has the form

$$R^l : \text{if } x_1 \text{ is } \hat{A}_1^l \text{ and} \ldots \text{and } x_n \text{is } \hat{A}_n^l \text{ then } y \text{ is } \hat{B}^l, \quad l = 1, \ldots, L. \quad (46)$$

If all of the antecedent and consequent T2 FSs are IT2 FSs, then we call the resulting T2 FLS an IT2 FLS. A rule-base contains four components: rules, fuzzifier, inference system, and output processing that consist of defuzzifier and type-reducer. The outputs of the T2 FLS are the type-reduced set and the crisp defuzzified value, [Mendel et al. 2009, 2007, and 2002].

4.1. Type-2 Singleton Fuzzification Model

From the rule (46), let $\hat{A}_1^l, \hat{A}_2^l, \ldots, \hat{A}_n^l$ be IT2 FSs in continuous universe of discourses $X_{1c}, X_{2c}, \ldots, X_{nc}$, respectively, and $\hat{B}^l$ be an IT2 FS in continuous universe of discourse Y_c. Decompose each $\hat{A}_i{}^l$ into its $n_{A_i} \to \infty$ $(i = 1, \ldots, n)$ embedded IT2 FSs $\hat{A}_{i_E}^{ki\,l}$, as the following:

$$\hat{A}_i^{\,l} = \int_{ki=1}^{n_{A_i}} A_{i_E}^{ki^{\,l}} = 1/\mathrm{DOU}\left(\hat{A}_i^{\,l}\right), \;\; i = 1, \dots, n., \tag{47}$$

where

$$\mathrm{DOU}\left(\hat{A}_i^{\,l}\right) = \int_{ki=1}^{n_{A_i}} A_{i_E}^{ki^{\,l}} = \int_{ki=1}^{n_{A_i}} \int_{j=1}^{N_{x_i}\to\infty} u_{ij}^{ki}/x_{ij}, \;\; u_{ij}^{ki} \in J_{x_{ij}} \subseteq [0,1]. \tag{48}$$

We also decompose $\hat{B}^l$ into $n_B \to \infty$ embedded IT2 FSs $\hat{B}_E^{k^{\,l}}$, whose domains are the embedded T1 FSs $B_E^{k^{\,l}}$; we see that $\hat{B}^l$ can be expressed as:

$$\hat{B}^l = \int_{k=1}^{n_B\to\infty} \hat{B}_E^{k^{\,l}} = 1/\mathrm{DOU}(\hat{B}^l) \tag{49}$$

where

$$\mathrm{DOU}(\hat{B}^l) = \int_{k=1}^{n_B\to\infty} B_E^{k^{\,l}} = \int_{k=1}^{n_B\to\infty} \int_{j=1}^{N_y\to\infty} v_j^k/y_j, \;\; v_j^k \in J_{y_j} \subseteq [0,1] \tag{50}$$

Cartesian product of antecedents, $\hat{A}_1^{\,l} \times \dots \times \hat{A}_n^{\,l}$ has $\left(\prod_{i=1}^{n} n_{A_i}\right) \to \infty$ collections of the embedded T1 FSs, $A_{i_E}^{ki^{\,l}}$. The relationship between $\hat{A}_i^{\,l}$ antecedents and consequent $\hat{B}^l$ can be represented by:

$$\mu_{\hat{A}_i^{\,l}\to\hat{B}^l}(\mathrm{x}, y) = \mu_{\hat{A}_1^{\,l}\times\dots\times\hat{A}_n^{\,l}\to\hat{B}^l}(\mathrm{x}, y) = \mu_{\hat{A}_1^{\,l}\times\dots\times\hat{A}_n^{\,l}}(\mathrm{x}) \dotplus \mu_{\hat{B}^l}(y)$$

$$= \mu_{\hat{A}_1^{\,l}}(x_1) \dotplus \dots \dotplus \mu_{\hat{A}_n^{\,l}}(x_n) \dotplus \mu_{\hat{B}^l}(y) = \left[S_{i=1}^{n} \mu_{\hat{A}_i^{\,l}}(x_i)\right] \dotplus \mu_{\hat{B}^l}(y), \tag{51}$$

where it has been supposed that Mamdani implications are used, multiple antecedents are connected by *or* (i.e. by S-norms), S is short for an S-norm and $\dotplus$ represents the max S-norms, [Zeng et al. 2008].

In general, there are L rules that describe an IT2 FLS and repeatedly more than one rule fires when input is applied to that system. Consequently, we have $n_{A_1} \times \dots \times n_{A_n} \times n_B$ collections of embedded T1 antecedent and consequent FSs, which generate all fired output sets for all collections of antecedent and consequent FSs, as the following, [Mendel et al. 2009, 2007, and 2002]:

$$D^l(y) = \int_{k1=1}^{n_{A_1}\to\infty} \dots \int_{kn=1}^{n_{A_n}\to\infty} \int_{k=1}^{n_B\to\infty} \mu_{D^l(k1,\dots,kn,k)}(y), \quad \forall\, y \in Y_c \tag{52}$$

The relationship between the consequent $D^l(y)$ in (52) and the DOU of the T2 fired output FS is made a summary by theorem 5a, [Wu and Mendel 2012, 2007 and 2002].

Theorem 4.1.15a: The output $D^l(y)$ in (52) that calculated by using T1 FS is the same as the DOU of the T2 fired output FS, which is calculated by using T2 FS.

Proof:

The fired output of the collection of the ki^{th} embedded T1 antecedents FS and the k^{th} embedded T1 consequent FS can be calculated for *SF* using Mamdani implication. If $A_i^{\,l}$ is a *type-2 FS*, the membership function is defined by equation (17):

$$\mu_{\hat{A}_i^{\,l}}(x_i) = \begin{Bmatrix} 1/0 & x_i \neq x_i' \\ 1/1 & x_i = x_i' \end{Bmatrix}, \tag{53}$$

then, from (51) and (53) we obtained:

$$\mu_{D^l(k1,\dots,kn,k)}(y) = \left[S_{i=1}^{n}\, \mu_{A_{i_E}^{ki^{\,l}}}(x_i')\right] \dotplus \mu_{B_E^{k^{\,l}}}(y), \;\; \forall\, y \in Y_c \;, \tag{54}$$

in which the bracketed term is often referred to as a *firing interval*, in which

$$S_{i=1}^{n}\, \mu_{A_{i_E}^{ki^{\,l}}}(x_i') = S_{i=1}^{n}\left[\underline{\mu}_{A_{i_E}^{ki^{\,l}}}(x_i'),\, \overline{\mu}_{A_{i_E}^{ki^{\,l}}}(x_i')\right]$$

$$= \left[S_{i=1}^{n}\, \underline{\mu}_{A_{i_E}^{ki^{\,l}}}(x_i'),\, S_{i=1}^{n}\, \overline{\mu}_{A_{i_E}^{ki^{\,l}}}(x_i')\right] = \left[\underline{a}(x'), \overline{a}(x')\right] = A(x') \tag{55}$$

Since, $\mu_{D^l(k1,\dots,kn,k)}(y)$ is limited by [0,1], then $D^l(y)$ in (52) also have to be a limited function in [0,1], and it contains an infinite and countable-infinite number of elements, so can be expressed $D^l(y)$ as:

$$D^l(y) = \left[\underline{\mu}_{\tilde{D}^l}(y), \dots, \overline{\mu}_{\tilde{D}^l}(y)\right], \qquad \forall\, y \in Y_c \tag{56}$$

Now, a collection of $n_{SF} = n_{A_1} \times \dots \times n_{A_n} \times n_B$ functions, where

$$\underline{\mu}_{\tilde{D}^l}(y) = \inf_{\forall\, k1,\dots,kn,k} \mu_{D^l(k1,\dots,kn,k)}(y), \quad \forall\, y \in Y_c \qquad (57)$$
$$\overline{\mu}_{\tilde{D}^l}(y) = \sup_{\forall\, k1,\dots,kn,k} \mu_{D^l(k1,\dots,kn,k)}(y), \quad \forall\, y \in Y_c \qquad (58)$$

Equations (57) and (58) denote the lower and upper bounding functions of $D^l(y)$, respectively.

Next, suppose $\underline{\mu}_{\hat{A}_i^l}(x_i)$ and $\overline{\mu}_{\hat{A}_i^l}(x_i)$ denote the lower and upper MFs for $\hat{A}_i^l$, thus $\underline{\mu}_{\tilde{B}^l}(y)$ and $\overline{\mu}_{\tilde{B}^l}(y)$ denote the lower and upper MFs for $\hat{B}^l$. As well as, suppose $\underline{\mu}_{A_i^l}(x_i)$ and $\overline{\mu}_{A_i^l}(x_i)$ denote the embedded T1 FSs associated with $\underline{\mu}_{\hat{A}_i^l}(x_i)$ and $\overline{\mu}_{\hat{A}_i^l}(x_i)$, respectively. We observe that $\underline{\mu}_{A_i^l}(x_i)$ and $\overline{\mu}_{A_i^l}(x_i)$ are two of the $(n_{A_i} \to \infty)$ embedded T1 FSs that are associated with $\hat{A}_i^l$. Also, $\underline{\mu}_{B^l}(y)$ and $\overline{\mu}_{B^l}(y)$denote the corresponding embedded T1 FSs of $\underline{\mu}_{\tilde{B}^l}(y)$ and $\overline{\mu}_{\tilde{B}^l}(y)$, respectively. Depending on (54), we tried to calculate the infimum of $\mu_{D^l(k1,\dots,kn,k)}(y)$ by choose the smallest embedded T1 FS of both the antecedent and consequent. Consequently, we obtain the following equation, [Mendel et al. 2009, 2006, and 2004]:

$$\underline{\mu}_{\tilde{D}^l}(y) = \left[\boldsymbol{S}_{i=1}^{n}\, \underline{\mu}_{A_i^l}(x_i')\right] \dot{+}\, \underline{\mu}_{B^l}(y), \qquad \forall\, y \in Y_c \qquad (59)$$

Similarly, we choose the largest embedded T1 FS of both the antecedent and consequent in order to compute the *supermom* of $\mu_{D^l(k1,\dots,kn,k)}(y)$, then, we obtain the following:

$$\overline{\mu}_{\tilde{D}^l}(y) = \left[\boldsymbol{S}_{i=1}^{n}\, \overline{\mu}_{A_i^l}(x_i')\right] \dot{+}\, \overline{\mu}_{B^l}(y), \qquad \forall\, y \in Y_c \qquad (60)$$

Comparing Equation (56) with (19), and from (55), we obtain:

$$D^l(y) = \mathrm{DOU}(\widehat{D}^l) = \left[\underline{\mu}_{\tilde{D}^l}(y), \overline{\mu}_{\tilde{D}^l}(y)\right] \qquad \forall\, y \in Y_c$$
$$= \left[\underline{a}^l(x') \dot{+} \underline{\mu}_{B^l}(y), \overline{a}^l(x') \dot{+} \overline{\mu}_{B^l}(y)\right] = \left[\underline{a}^l(x'), \overline{a}^l(x')\right] \dot{+} \left[\underline{\mu}_{B^l}(y), \overline{\mu}_{B^l}(y)\right], \qquad \forall\, y \in Y_c \qquad (61)$$

Moreover, from (18) we derive that, $\widehat{D}^l = 1/\mathrm{DOU}(\widehat{D}^l)$. Then, we have been able to obtain the DOU of the T2 fired output FS using T1 FS mathematics.

4.2. Type-2 Non-singleton Fuzzification Model

Let the n-dimensional input is given by the IT2 FS, and we suppose $\hat{X}_i$ denote the IT2 FSs describing each of the n inputs. More specifically, $\hat{X}_1, \hat{X}_2, \dots, \hat{X}_n$ are IT2 FSs in continuous universes of discourse $X_{1c}, X_{2c}, \dots, X_{nc}$. There are L rules that described an IT2 FLS, and repeatedly more than one rule fires when input is applied to that system. Decompose $\hat{A}_i$ into their $n_{X_i} \to \infty$ $(i = 1, \dots, n)$ embedded IT2 FSs $\hat{X}_{i_E}^{hi}$, i.e., [Castro et al. 2008], [Mendel et al. 2009, 2007, and 2002],

$$\hat{X}_i = \int_{hi=1}^{n_{X_i} \to \infty} \hat{X}_{i_E}^{hi}, \qquad i = 1, \dots, n. \qquad (62)$$

The domain of each $\hat{X}_{i_E}^{hi}$ is the embedded T1 FS $X_{i_E}^{hi}$. The Cartesian product be $\hat{X}_1 \times \hat{X}_2 \times \dots \times \hat{X}_n$, has $\left(\prod_{i=1}^{n} n_{X_i}\right) \to \infty$ collections of the embedded T1 FSs $X_{i_E}^{hi}$, then the MF of a fuzzy Cartesian product is given by:

$$\mu_{X_1}(x_1) \dot{+} \dots \dot{+} \mu_{X_n}(x_1) = \boldsymbol{S}_{i=1}^{n}\, \mu_{X_i}(x_i) \qquad (63)$$

Since, each rule determines a fuzzy set D in Y such that when we use Zadeh's sup-star composition, note that:

$$\mu_{D^l}(y) = \sup_{\mathrm{x}, x_i \in X_c} \left[\left(\mu_{X_1}(x_1) \dot{+} \dots \dot{+} \mu_{X_n}(x_n)\right) \dot{+} \mu_{\hat{A}_i^l \to B^l}(\mathrm{x}, y)\right], \qquad y \in Y$$
$$= \sup_{\mathrm{x}, x_i \in X_c} \left[\boldsymbol{S}_{i=1}^{n}\, \mu_{X_i}(x_i) \dot{+} \boldsymbol{S}_{i=1}^{n}\, \mu_{\hat{A}_i^l}(x_i) \dot{+} \mu_{B^l}(y)\right] = \sup_{\mathrm{x}, x_i \in X_c} \left\{\left[\boldsymbol{S}_{i=1}^{n}\, \mu_{X_i}(x_i) \dot{+}\ \mu_{\hat{A}_i^l}(x_i)\right] \dot{+} \mu_{B^l}(y)\right\} \qquad (64)$$

Then, we have derived the formula of *NSF* as the following:

$$\mu_{D^l}(y) = \left[\boldsymbol{S}_{i=1}^{n} \left(\sup_{x_i \in X_{ic}} \mu_{X_i}(x_i) \dot{+}\ \mu_{\hat{A}_i^l}(x_i)\right)\right] \dot{+} \mu_{B^l}(y), \qquad \forall\ y \in Y_c \qquad (65)$$

Since, there are $n_B \to \infty$ embedded T1 FSs for the consequent, $\left(n_A = \prod_{i=1}^{n} n_{A_i}\right) \to \infty$ embedded T1 FSs for the antecedents, and $\left(n_X = \prod_{i=1}^{n} n_{X_i}\right) \to \infty$ embedded T1 FSs for the inputs; then, we obtain $(n_{X_1} \times \dots \times n_{X_n} \times n_{A_1} \times \dots \times n_{A_n} \times n_B)$ collections of input, antecedent, and consequent embedded T1 FSs as shown in Fig. 4, which generate $\mu_{D^l}(y)$ as the following, [Mendel et al. 2009, 2006, and 2004]:

$$\mu_{D^l}(y) = \int_{h_1=1}^{n_{X_1} \to \infty} \dots \int_{h_n=1}^{n_{X_n} \to \infty} \int_{k_1=1}^{n_{A_1} \to \infty} \dots \int_{k_n=1}^{n_{A_n} \to \infty} \int_{k=1}^{n_B \to \infty} \mu_{D^l(h_1,\dots,h_n,k_1,\dots,k_n,k)}(y) \qquad (66)$$

In order to represent the structure of $\mu_{D^l}(y)$ using network, we depict (66) through Fig. 4, for a *single-antecedent rule*.

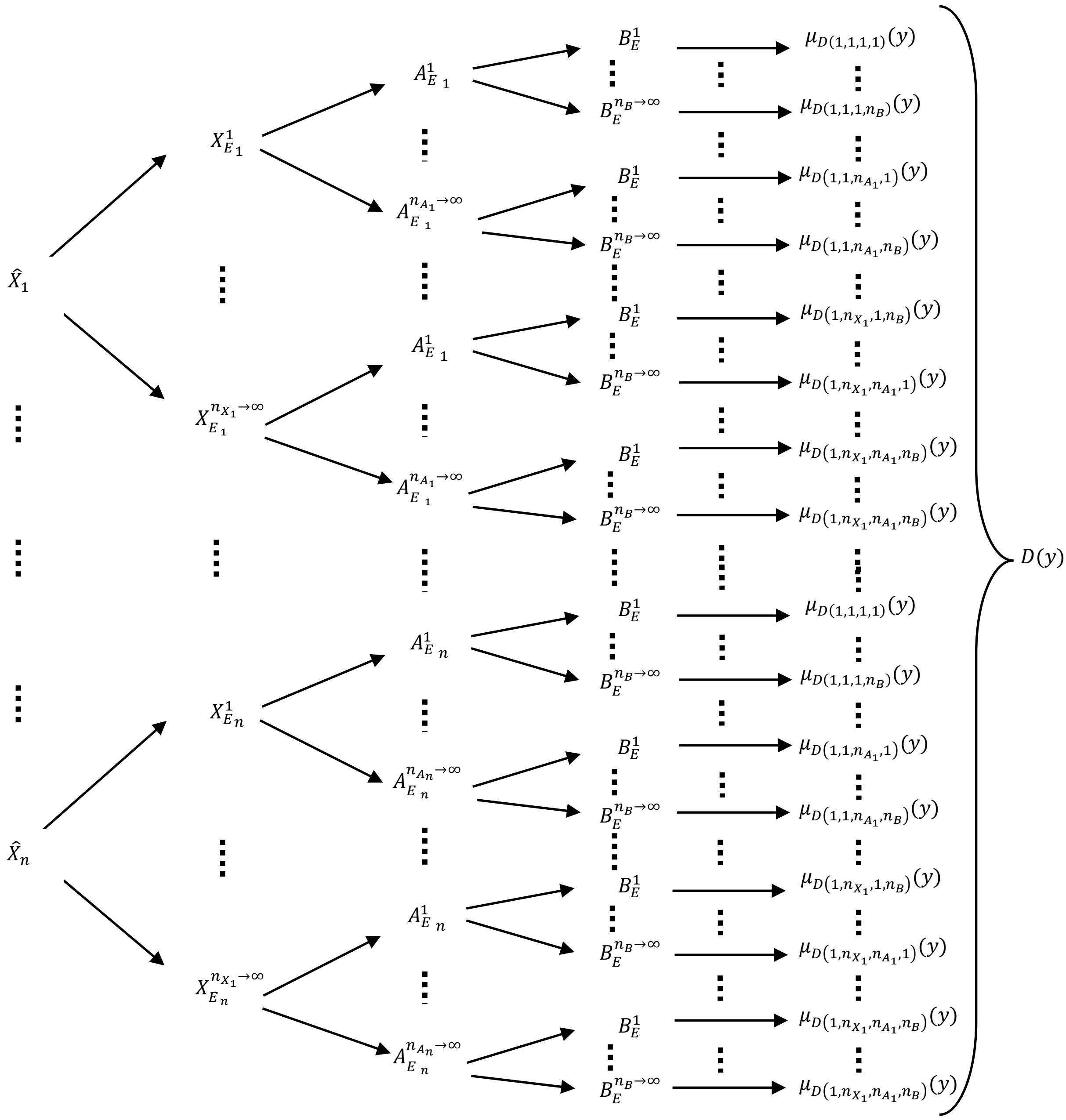


Fig. 4. Fired output FSs $\forall\, n_D = n_X \times n_A \times n_B$ collections of the embedded T1 antecedent and consequent FSs for N Antecedent rules .

Theorem 4.2.1 3b: The output $D^l(y)$ in (66), calculated by using T1 FS is the same as the DOU of the T2 fired output FS, which is calculated by using T2 FS.

Proof:
The fired output of the collection of the hi^{th} embedded T1 antecedents FS and the h^{th} embedded T1 consequent FS can be calculated for NSF using Mamdani implication. Depending on a formula (65) of *NSF*, we could calculate $\mu_{D^l}(y)$ as the following:

$$\mu_{D^l}(y) = \int_{h_1=1}^{n_{X_1\to\infty}} \dots \int_{h_n=1}^{n_{X_n\to\infty}} \int_{k_1=1}^{n_{A_1\to\infty}} \dots \int_{k_n=1}^{n_{A_n\to\infty}} \int_{k=1}^{n_{B}\to\infty} \left[\boldsymbol{S}_{i=1}^{n} \left(\sup_{x_i \in X_{ic}} \mu_{X_i}(x_i) \dotplus \mu_{A_i{}^l}(x_i) \right) \right] \dotplus \mu_{B^l}(y),$$

$$= \int_{h_i=1}^{n_{X_i\to\infty}} \int_{k_i=1}^{n_{A_i\to\infty}} \int_{k=1}^{n_B\to\infty} \left[\boldsymbol{S}_{i=1}^{n} \left(\sup_{x_i\in X_{ic}} \mu_{X_i}(x_i) \dotplus \mu_{A_i{}^l}(x_i) \right) \right] \dotplus \mu_{B^l}(y), \quad \forall\ y \in Y_c \tag{67}$$

Note that $\mu_{D^l(h_1,\ldots,h_n,k_1\ldots,k_n,k)}(y)$ is limited by [0,1], then $D^l(y)$ in (66) also must be a limited function in [0,1], and it contains an infinite and countable-infinite number of elements, so it can be expressed $D^l(y)$ as:

$$D^l(y) = \left[\underline{\mu}_{\tilde{D}^l}(y), \ldots, \overline{\mu}_{\tilde{D}^l}(y) \right] \qquad \forall\ y \in Y_c \tag{68}$$

Now, a collection of $\left(n_{NSF} = n_{X_1} \times \ldots \times n_{X_n} \times n_{A_1} \times \ldots \times n_{A_n} \times n_B\right) \to \infty$ functions, where

$$\underline{\mu}_{\tilde{D}^l}(y) = \inf_{\forall\ h_1,\ldots,h_n,k_1\ldots,k_n,k} \mu_{D^l(h_1,\ldots,h_n,k_1\ldots,k_n,k)}(y), \quad \forall\ y \in Y_c \tag{69}$$

$$\overline{\mu}_{\tilde{D}^l}(y) = \sup_{\forall\ h_1,\ldots,h_n,k_1\ldots,k_n,k} \mu_{D^l(h_1,\ldots,h_n,k_1\ldots,k_n,k)}(y), \quad \forall\ y \in Y_c \tag{70}$$

Equations (69) and (70) denote the lower and upper bound functions of $D^l(y)$, respectively.

Next, suppose $\underline{\mu}_{\tilde{X}_i}(x_i)$ and $\overline{\mu}_{\tilde{X}_i}(x_i)$ denote the lower and upper MFs for $\hat{X}_i$. Additionally, suppose $\underline{\mu}_{X_i}(x_i)$ and $\overline{\mu}_{X_i}(x_i)$ denote the embedded T1 FSs associated with $\underline{\mu}_{\tilde{X}_i}(x_i)$ and $\overline{\mu}_{\tilde{X}_i}(x_i)$, respectively. We observe that $\underline{\mu}_{X_i}(x_i)$ and $\overline{\mu}_{X_i}(x_i)$ are two of the $\left(n_{X_i} \to \infty\right)$ embedded T1 FSs that are associated with $\hat{X}_i$. For $\hat{A}_i{}^l$, suppose $\underline{\mu}_{\hat{A}_i{}^l}(x_i)$ and $\overline{\mu}_{\hat{A}_i{}^l}(x_i)$ denote the lower and upper MFs for $\hat{A}_i{}^l$, therefore $\underline{\mu}_{\tilde{B}^l}(y)$ and $\overline{\mu}_{\tilde{B}^l}(y)$ denote the lower and upper MFs for $\hat{B}^l$. As well as, suppose $\underline{\mu}_{A_i{}^l}(x_i)$ and $\overline{\mu}_{A_i{}^l}(x_i)$ denote the embedded T1 FSs associated with $\underline{\mu}_{\hat{A}_i{}^l}(x_i)$ and $\overline{\mu}_{\hat{A}_i{}^l}(x_i)$, respectively. We observe that $\underline{\mu}_{A_i{}^l}(x_i)$ and $\overline{\mu}_{A_i{}^l}(x_i)$ and are two of the $\left(n_{A_i} \to \infty\right)$ embedded T1 FSs that are associated with $\hat{A}_i{}^l$. Also, $\underline{\mu}_{B^l}(y)$ and $\overline{\mu}_{B^l}(y)$ denote the corresponding embedded T1 FSs of $\underline{\mu}_{\tilde{B}^l}(y)$ and $\overline{\mu}_{\tilde{B}^l}(y)$, respectively. Depending on (65), we tried to calculate the infimum of $\mu_{D^l(h_1,\ldots,h_n,k_1\ldots,k_n,k)}(y)$ by choosing the smallest embedded T1 FS of both the antecedent and consequent. Consequently, we obtain the following equation, [Mendel et al. 2009, 2006, and 2001]:

$$\underline{\mu}_{\tilde{D}^l}(y) = \inf_{\forall\ h_i,k_i,k} \mu_{D^l(h_i,k_i,k)}(y) = \left[\boldsymbol{S}_{i=1}^{n} \left(\sup_{x_i\in X_{ic}} \underline{\mu}_{X_i}(x_i) \dotplus \underline{\mu}_{A_i{}^l}(x_i) \right) \right] \dotplus \underline{\mu}_{B^l}(y), \quad \forall\ y \in Y_c \tag{71}$$

Similarly, we choose the largest embedded T1 FS of both the antecedent and consequent in order to compute the *supermom* of $\mu_{D^l(h_1,\ldots,h_n,k_1\ldots,k_n,k)}(y)$, then we obtain the following equation:

$$\overline{\mu}_{\tilde{D}^l}(y) = \sup_{\forall\ h_i,k_i,k} \mu_{D^l(h_i,k_i,k)}(y) = \left[\boldsymbol{S}_{i=1}^{n} \left(\sup_{x_i\in X_{ic}} \overline{\mu}_{X_i}(x_i) \dotplus \overline{\mu}_{A_i{}^l}(x_i) \right) \right] \dotplus \overline{\mu}_{B^l}(y), \quad \forall\ y \in Y_c \tag{72}$$

Comparing the Equation (68) with (19), we obtain:

$$D^l(y) = \mathrm{DOU}(\widehat{D}^l) = \left[\underline{\mu}_{\tilde{D}^l}(y), \overline{\mu}_{\tilde{D}^l}(y) \right], \qquad \forall\ y \in Y_c \tag{73}$$

Moreover, from (18) we derive that

$$\widehat{D}^l = 1/\mathrm{DOU}(\widehat{D}^l) \tag{74}$$

Consequently, we have been able to obtain the DOU of the T2 fired output FS using T1 FS mathematics.

5. THE OUTPUT PROCESSING

Type-Reduction (TR) is a first step of output processing, in order to compute the centroid of an IT2 FS. We are derived to compute the centroid of an IT2 FS because when all sources of uncertainty disappear, the IT2 FLS must reduce to a T1 FLS. We define the centroid $(C_{\tilde{D}})$ of an IT2 FS $\widehat{D}$ such as the set of the centroids of all of its embedded IT2 FSs. depending on (18) and (19) note that, we must compute the centroid of all of the $n_D \to \infty$ embedded T1 FSs contained within $\mathrm{DOU}(\widehat{D}^l)$. Therefore, we obtain a set of n_D numbers that have both a minimum and maximum element, $c_l(\widehat{D}) \equiv c_l$ and $c_r(\widehat{D}) \equiv c_r$, respectively. The centroid of each of the embedded T1 FSs is a limited number. Related with each of these numbers will be a membership degree of one as the following, [Karnik et al. 2001], [Wu and Mendel 2012, 2007 and 2002], and [Salazar et al 2011]:

$$C_{\tilde{D}} = 1/\left[c_l(\widehat{D}), c_r(\widehat{D})\right]$$

The generalized centroid $\left[c_l(\widehat{D}), c_r(\widehat{D})\right]$ is a closed interval, c_l and c_r can be computed from the lower and upper MF of $\hat{A}$ as follows:

$$c_l(\widehat{D}) = \min\{\text{centroid of all embedded T1 FSs in } \mathrm{DOU}(\widehat{D})\} = \min_{l\in R}\left(C(A_{E_l})\right)$$

$$c_l(\widehat{D}) = \min_{l\in R}\left(\frac{\int_1^{\infty} x\,\mu_{A_{E_l}}(x)\,dx}{\int_1^{\infty}\mu_{A_{E_l}}(x)\,dx}\right) = \min_{l\in R}\left(\frac{\int_1^{l} x\,\overline{\mu}_{\hat{A}}(x)\,dx + \int_{l+1}^{\infty} x\,\underline{\mu}_{\hat{A}}(x)\,dx}{\int_1^{l}\overline{\mu}_{\hat{A}}(x)\,dx + \int_{l+1}^{\infty}\underline{\mu}_{\hat{A}}(x)\,dx}\right) \tag{75}$$

$c_r(\widehat{D})$ = max {centroid of all embedded T1 FSs in DOU$(\widehat{D})$}= $\max_{r\in R}\left(C(A_{E_r})\right)$

$$c_r(\widehat{D}) = \max_{r\in R}\left(\frac{\int_1^{\infty} x\,\mu_{A_{E_r}}(x)\,dx}{\int_1^{\infty}\mu_{A_{E_r}}(x)\,dx}\right) = \max_{r\in R}\left(\frac{\int_1^{r} x\,\underline{\mu}_{\hat{A}}(x)\,dx + \int_{r+1}^{\infty} x\,\overline{\mu}_{\hat{A}}(x)\,dx}{\int_1^{r}\underline{\mu}_{\hat{A}}(x)\,dx + \int_{r+1}^{\infty}\overline{\mu}_{\hat{A}}(x)\,dx}\right), \tag{76}$$

in which $\mu_{A_{E_l}}$ and $\mu_{A_{E_r}}$ denote embedded type-1 fuzzy sets as the following:

$$\mu_{A_{E_l}}(x) = \begin{cases}\overline{\mu}_{\hat{A}}(x), & if\ x \le l,\\ \underline{\mu}_{\hat{A}}(x), & if\ x > l,\end{cases} \tag{77a}$$

$$\mu_{A_{E_l}}(x) = \begin{cases}\underline{\mu}_{\hat{A}}(x), & if\ x \le r,\\ \overline{\mu}_{\hat{A}}(x), & if\ x > r.\end{cases} \tag{77b}$$

where $l, r(\in X_c \subseteq R)$ are switch points that mark the change from $\overline{\mu}_{\hat{A}}(x)$ to $\underline{\mu}_{\hat{A}}(x)$ and from $\underline{\mu}_{\hat{A}}(x)$ to $\overline{\mu}_{\hat{A}}(x)$, respectively. $\underline{\mu}_{\hat{A}}(x)$ and $\overline{\mu}_{\hat{A}}(x)$ are respectively the lower and upper membership functions of $\hat{A}$. See Fig. 5., Salazar et al. 2011. There are different kinds of type-reduction as center-of-sums, height, and center-of-sets type-reducers.

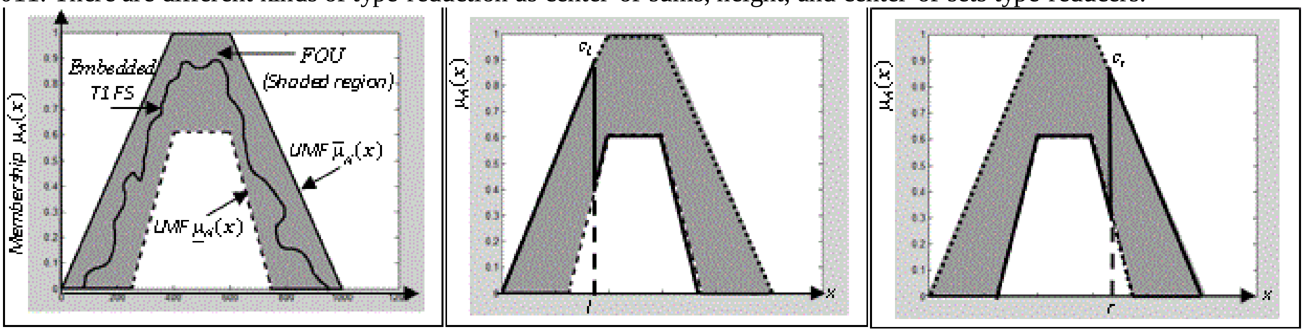


Fig. 5. (a) Interval Type-2 fuzzy set (b) Explanation of the switch point l (c) Explanation of the switch

5.1. Derivation of Type-Reduction for Interval T2 FLS

The general form for continuous domain in order to calculate the different kinds of type-reduced can all be given by, [Mendel et al. 2009, 2007 and 2002], [Salazar et al. 2011]:

$$Y_{TR}(\mathbb{Y}^1, \dots, \mathbb{Y}^{\infty}, \mathbb{A}^1, \dots, \mathbb{A}^{\infty}) = \int_{y^1\in[y_l^1, y_r^1]} \cdots \int_{y^{\infty}\in[y_l^{\infty}, y_r^{\infty}]} \int_{a^1\in[\underline{a}^1, \overline{a}^1]} \cdots \int_{a^{\infty}\in[\underline{a}^{\infty}, \overline{a}^{\infty}]} 1 \Big/ \frac{\int_{i=1}^{L=\infty} y^i\, a^i}{\int_{i=1}^{L=\infty} a^i}, \tag{78}$$

where each one of $y_l^i, y_r^i, \underline{a}^i\ \overline{a}^i$ $(i = 1, \dots, \infty)$ and L have various meaning, as follows, [Mendel 2004]:

1. In case of centroid and center of sums be $y_l^i = y_r^i$, the i^{th} point in the sampled universe of discourse of the FLS's output, $\underline{a}^i, \overline{a}^i$ be the single (or sums of all rules for COS) of lower and upper membership degrees for the i^{th} sampled point; contains antecedent and consequent MF parameters, and L is a number of sampled points.
2. In case of center of sets be y_l^i, y_r^i left and right endpoints of the centroid of the consequent of i^{th} rule. While with height TR is $y_l^i = y_r^i$, a single point in the consequent domain of i^{th} rule, treated as a consequent parameter. In the COS and height TR, $\underline{a}^i, \overline{a}^i$ be lower and upper firing degrees for the i^{th} rule; contains antecedent MF parameters, and L is a number of rules.

Because all the memberships in an interval type-1 set are crisp then we represent an interval set by its domain interval. It can be represented by its center and spread as $[c - s, c + s]$, where $c = (r + l)/2$ and $s = (r - l)/2$, where l left and r right endpoints. Each $\mathbb{Y}^i$ in (78) is an IT1 S having center $c_{\mathbb{Y}}^i$ and spread $s_{\mathbb{Y}}^i \ge 0$. Each $\mathbb{A}^i$ is also IT1 S with center $c_{\mathbb{A}}^i$ and spread $s_{\mathbb{A}}^i \ge 0$ (suppose $c_{\mathbb{A}}^i \ge s_{\mathbb{A}}^i, \forall\, i = 1, \dots, \infty$). Therefore, we need to calculate its two endpoints $[y_l, y_r]$.

Now, we assume that,

$$S(a^i, \dots, a^{L=\infty}) = \frac{\int_{i=1}^{L=\infty} y^i\, a^i}{\int_{i=1}^{L=\infty} a^i} \tag{79}$$

where $a^i \in [c_{\mathbb{A}}^i - s_{\mathbb{A}}^i, c_{\mathbb{A}}^i + s_{\mathbb{A}}^i]$ and $y^i \in [c_{\mathbb{Y}}^i - s_{\mathbb{Y}}^i, c_{\mathbb{Y}}^i + s_{\mathbb{Y}}^i]$. Next, we explain an iterative procedure to compute left endpoint, $y_l = \min(S)$, and right endpoint, $y_r = \max(S)$, for IT2 FLS.

In order to compute $(\min(S))$, we put $y^i = c_{\mathbb{Y}}^i - s_{\mathbb{Y}}^i$ $(i = 1, \dots, \infty)$ and suppose $y^1 \le y^2 \le \dots \le y^\infty$, therefore,

a) Put $a^i = c_{\mathbb{A}}^i$, $i = 1, \dots, \infty$ and calculated $S' = S(c_{\mathbb{A}}^1, \dots, c_{\mathbb{A}}^\infty)$ by using (79);

b) Find $1 \le h \le \infty$ s.t. $y^h \le S' \le y^{h+e}$, where e is a so small real number;

c) We put $a^i = c_{\mathbb{A}}^i + s_{\mathbb{A}}^i$, $\forall\ i \in [1, h]$ and $a^i = c_{\mathbb{A}}^i - s_{\mathbb{A}}^i$, $\forall\, i \in [h+e, \infty)$, thus compute S'' using (79) as follows:

$$S'' = S\left((c_{\mathbb{A}}^1 + s_{\mathbb{A}}^1), \dots, (c_{\mathbb{A}}^h + s_{\mathbb{A}}^h), (c_{\mathbb{A}}^{h+e} - s_{\mathbb{A}}^{h+e}), \dots, (c_{\mathbb{A}}^\infty - s_{\mathbb{A}}^\infty)\right) \tag{80}$$

d) If $S'' = S'$, then stop and put S'' is the minimum value of S; else continue.

e) Put $S' = S''$ go back to step b.

For compute $(\max(S))$, we put $y^i = c_{\mathbb{Y}}^i + s_{\mathbb{Y}}^i$ $(i = 1, \dots, \infty)$, and suppose $y^1 \le y^2 \le \dots \le y^\infty$. Therefore

a) Put $a^i = c_{\mathbb{A}}^i$, $i = 1, \dots, \infty$ and calculate $S' = S(c_{\mathbb{A}}^1, \dots, c_{\mathbb{A}}^\infty)$ by using (79);

b) Find h $(1 \le h \le \infty)$ s.t. $y^h \le S' \le y^{h+e}$, where e is a very small real number;

c) We put $a^i = c_{\mathbb{A}}^i - s_{\mathbb{A}}^i$, $\forall\ i \in [1, h]$ and $a^i = c_{\mathbb{A}}^i + s_{\mathbb{A}}^i$, $\forall\, i \in [h+e, \infty)$, thus compute S'' using (79) as follows:

$$S'' = S\left((c_{\mathbb{A}}^1 - s_{\mathbb{A}}^1), \dots, (c_{\mathbb{A}}^h - s_{\mathbb{A}}^h), (c_{\mathbb{A}}^{h+e} + s_{\mathbb{A}}^{h+e}), \dots, (c_{\mathbb{A}}^\infty + s_{\mathbb{A}}^\infty)\right) \tag{81}$$

d) If $S'' = S'$, then stop and put S'' is the maximum value of S; else continue;

e) Put $S' = S''$ go back to step b.

This procedure of computational can be used to calculate the TR set for all of the kind reducers, with a great reduction in computational complexity.

5.2. Defuzzification

Since Y_{TR} is an interval set for all kinds of type-reduction method, we defuzzify it using the average of y_l and y_r, [Mendel et al. 2009, 2007 and 2002], therefore, the defuzzified output of IT2 FLS is

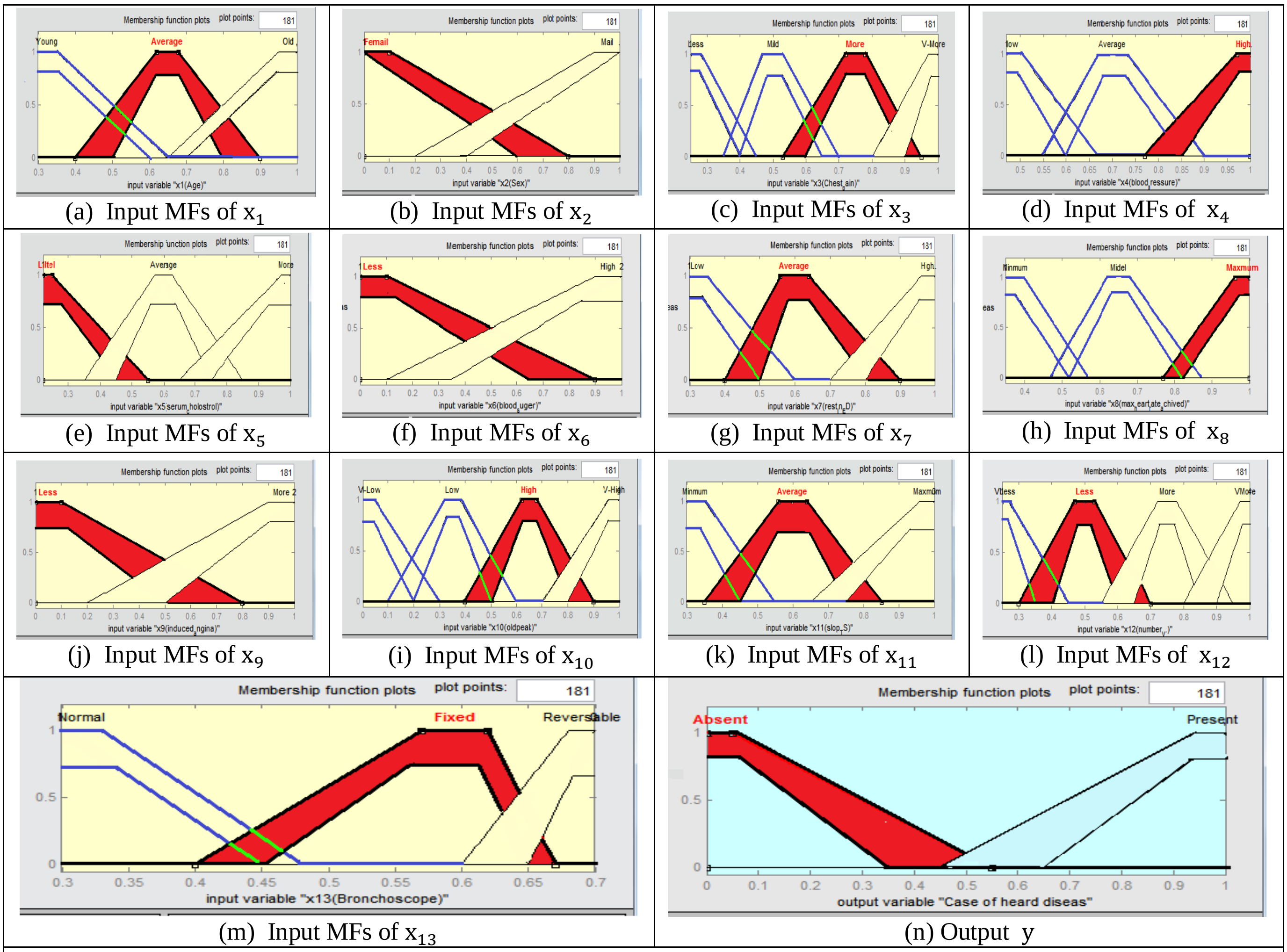
$$y(\mathrm{x}) = \frac{c_l + c_r}{2} \tag{82}$$

6. APPLICATION OF AN IT2 FLS

The purpose of this section is to provide real medical application for the IT2 *FLS*. The mathematical operations in an IT2 *FLS* are explained using an application. This application cares about the heart diseases, where we are able to determine the status of the heart for the people who suffer from heart disease or not, that depend a range of analyzes and tests performed for each person. Application of the heart disease contains thirteen attributes (which have been extracted from a larger set of 75): Age; Sex; Chest pain type; Resting blood pressure; Serum cholesterol in mg/dl; Fasting blood sugar; Resting electrocardiographic results; Maximum heart rate achieved; Exercise induced angina; Old peak (ST depression induced by exercise relative to rest); The slope of the peak exercise ST segment; Number of major vessels colored by fluoroscopy; and Thal: Normal, fixed defect, and reversible defect. This data was obtained from “StatLib. http://datamarket.com/data/set/22vj/”. The prediction variable (output) is an absence or presence of heart disease. There are 270 observations, and no missing values.

6.1. Discussion and Simulation Results

Consider an IT2 FLS that has thirteen inputs $(x_1, x_2, \dots, x_{13})$ and one output y. Each input domain consists of maximum four IT2 FSs, which are trapezoids LMFs and UMFs, and theirs FOUs shown in Fig. 6.



(a) Input MFs of x_1 (b) Input MFs of x_2 (c) Input MFs of x_3 (d) Input MFs of x_4

(e) Input MFs of x_5 (f) Input MFs of x_6 (g) Input MFs of x_7 (h) Input MFs of x_8

(j) Input MFs of x_9 (i) Input MFs of x_{10} (k) Input MFs of x_{11} (l) Input MFs of x_{12}

(m) Input MFs of x_{13} (n) Output y

Fig. 6. The IT2 FSs of heart diseases inputs and output MFs that have different uncertainty

The rule-base of the IT2 FLS has multi models for rules as the following:

R^1: if x_1 *is* $\hat{A}_{1,3} \wedge x_2$ *is* $\hat{A}_{2,1} \wedge x_3$ *is* $\hat{A}_{3,3} \wedge x_4$ *is* $\hat{A}_{4,1} \wedge x_5$ *is* $\hat{A}_{5,3} \wedge x_6$ *is* $\hat{A}_{6,1} \wedge x_7$ *is* $\hat{A}_{7,3} \wedge x_8$ *is* $\hat{A}_{8,2} \wedge x_9$ *is* $\hat{A}_{9,1} \wedge x_{10}$ *is* $\hat{A}_{10,2} \wedge$
x_{11} *is* $\hat{A}_{11,2} \wedge x_{12}$ *is* $\hat{A}_{12,1} \wedge x_{13}$ *is* $\hat{A}_{13,3}$ then y *is* Y^1

R^2: if x_1 *is* $\hat{A}_{1,2} \wedge x_2$ *is* $\hat{A}_{2,1} \wedge x_3$ *is* $\hat{A}_{3,2} \wedge x_4$ *is* $\hat{A}_{4,1} \wedge x_5$ *is* $\hat{A}_{5,1} \wedge x_6$ *is* $\hat{A}_{6,1} \wedge x_7$ *is* $\hat{A}_{7,1} \wedge x_8$ *is* $\hat{A}_{8,3} \wedge x_9$ *is* $\hat{A}_{9,1} \wedge x_{10}$ *is* $\hat{A}_{10,1} \wedge$
x_{11} *is* $\hat{A}_{11,1} \wedge x_{12}$ *is* $\hat{A}_{12,1} \wedge x_{13}$ *is* $\hat{A}_{13,1}$ then y *is* Y^1

R^3: if x_1 *is* $\hat{A}_{1,2} \wedge x_2$ *is* $\hat{A}_{2,2} \wedge x_3$ *is* $\hat{A}_{3,4} \wedge x_4$ *is* $\hat{A}_{4,2} \wedge x_5$ *is* $\hat{A}_{5,2} \wedge x_6$ *is* $\hat{A}_{6,1} \wedge x_7$ *is* $\hat{A}_{7,3} \wedge x_8$ *is* $\hat{A}_{8,3} \wedge x_9$ *is* $\hat{A}_{9,2} \wedge x_{10}$ *is* $\hat{A}_{10,1} \wedge$
x_{11} *is* $\hat{A}_{11,1} \wedge x_{12}$ *is* $\hat{A}_{12,1} \wedge x_{13}$ *is* $\hat{A}_{13,2}$ then y *is* Y^1

R^4: if x_1 *is* $\hat{A}_{1,2} \wedge x_2$ *is* $\hat{A}_{2,2} \wedge x_3$ *is* $\hat{A}_{3,4} \wedge x_4$ *is* $\hat{A}_{4,1} \wedge x_5$ *is* $\hat{A}_{5,2} \wedge x_6$ *is* $\hat{A}_{6,2} \wedge x_7$ *is* $\hat{A}_{7,1} \wedge x_8$ *is* $\hat{A}_{8,2} \wedge x_9$ *is* $\hat{A}_{9,1} \wedge x_{10}$ *is* $\hat{A}_{10,1} \wedge$
x_{11} *is* $\hat{A}_{11,1} \wedge x_{12}$ *is* $\hat{A}_{12,4} \wedge x_{13}$ *is* $\hat{A}_{13,3}$ then y *is* Y^1

R^5: if x_1 *is* $\hat{A}_{1,3} \wedge x_2$ *is* $\hat{A}_{2,2} \wedge x_3$ *is* $\hat{A}_{3,1} \wedge x_4$ *is* $\hat{A}_{4,2} \wedge x_5$ *is* $\hat{A}_{5,2} \wedge x_6$ *is* $\hat{A}_{6,2} \wedge x_7$ *is* $\hat{A}_{7,3} \wedge x_8$ *is* $\hat{A}_{8,3} \wedge x_9$ *is* $\hat{A}_{9,1} \wedge x_{10}$ *is* $\hat{A}_{10,2} \wedge$
x_{11} *is* $\hat{A}_{11,2} \wedge x_{12}$ *is* $\hat{A}_{12,1} \wedge x_{13}$ *is* $\hat{A}_{13,2}$ then y *is* Y^2

R^6: if x_1 *is* $\hat{A}_{1,2} \wedge x_2$ *is* $\hat{A}_{2,2} \wedge x_3$ *is* $\hat{A}_{3,4} \wedge x_4$ *is* $\hat{A}_{4,2} \wedge x_5$ *is* $\hat{A}_{5,1} \wedge x_6$ *is* $\hat{A}_{6,1} \wedge x_7$ *is* $\hat{A}_{7,3} \wedge x_8$ *is* $\hat{A}_{8,2} \wedge x_9$ *is* $\hat{A}_{9,2} \wedge x_{10}$ *is* $\hat{A}_{10,2} \wedge$
x_{11} *is* $\hat{A}_{11,2} \wedge x_{12}$ *is* $\hat{A}_{12,3} \wedge x_{13}$ *is* $\hat{A}_{13,3}$ then y *is* Y^2

:

R^n: if x_1 *is* $\hat{A}_{1,1} \wedge x_2$ *is* $\hat{A}_{2,2} \wedge x_3$ *is* $\hat{A}_{3,4} \wedge x_4$ *is* $\hat{A}_{4,2} \wedge x_5$ *is* $\hat{A}_{5,1} \wedge x_6$ *is* $\hat{A}_{6,1} \wedge x_7$ *is* $\hat{A}_{7,1} \wedge x_8$ *is* $\hat{A}_{8,2} \wedge x_9$ *is* $\hat{A}_{9,2}$
$\wedge\, x_{10}$ *is* $\hat{A}_{10,2} \wedge$
x_{11} *is* $\hat{A}_{11,2} \wedge x_{12}$ *is* $\hat{A}_{12,1} \wedge x_{13}$ *is* $\hat{A}_{13,3}$ then y *is* Y^2

There are 270 observations of heart diseases divided into two parts: the first part contains 150 cases were the heart disease absence, and the second part contains 120 cases were the heart disease presence. Each case that has thirteen an inputs are described by IT2 FSs. Consider one case of an input vector,
$x' = (x_1', x_2', x_3', x_4', x_5', x_6', x_7', x_8', x_9', x_{10}', x_{11}', x_{12}', x_{13}') =$
$(0.87, 0.01, 0.78, 0.55, 0.95, 0.14, 0.94, 0.79, 0.14, 0.26, 0.6, 0, 0.27, 0.69)$.
The firing intervals for the first fifth rules at an input vector x' are as the following (see Table 1):

Table 1. The firing intervals for the rules at an input vector x'

No	R^1	R^2	R^3	R^4	R^5	R^6	..	R^n
x_1	$[\mu_{\underline{A}_{1,3}}(x_1'),\mu_{\bar{A}_{1,3}}($ $= [0.56,0.73]$	$[\mu_{\underline{A}_{1,2}}(x_1'),\mu_{\bar{A}_{1,2}}$ $= [0.6,0.86]$	$[\mu_{\underline{A}_{1,2}}(x_1'),\mu_{\bar{A}_{1,2}}$ $= [0.8,1]$	$[\mu_{\underline{A}_{1,2}}(x_1'),\mu_{\bar{A}_{1,2}}$ $= [0.8,1]$	$[\mu_{\underline{A}_{1,3}}(x_1'),\mu_{\bar{A}_{1,3}}$ $= [0.13,0.55]$	$[\mu_{\underline{A}_{1,2}}(x_1'),\mu_{\bar{A}_{1,2}}$ $= [0.47,0.63]$	..	$[\mu_{\underline{A}_{1,1}}(x_1'),\mu_{\bar{A}_{1,1}}$ $= [0.5,0.67]$
x_2	$[\mu_{\underline{A}_{2,1}}(x_2'),\mu_{\bar{A}_{2,1}}($ $= [0.8,0.998]$	$[\mu_{\underline{A}_{2,1}}(x_2'),\mu_{\bar{A}_{2,1}}$ $= [0.98,1]$	$[\mu_{\underline{A}_{2,2}}(x_2'),\mu_{\bar{A}_{2,2}}$ $= [0.83,1]$	$[\mu_{\underline{A}_{2,2}}(x_2'),\mu_{\bar{A}_{2,2}}$ $= [0.83,1]$	$[\mu_{\underline{A}_{2,2}}(x_2'),\mu_{\bar{A}_{2,2}}$ $= [0.98,1]$	$[\mu_{\underline{A}_{2,2}}(x_2'),\mu_{\bar{A}_{2,2}}$ $= [0.83,1]$	..	$[\mu_{\underline{A}_{2,2}}(x_2'),\mu_{\bar{A}_{2,2}}$ $= [0.83,1]$
x_3	$[\mu_{\underline{A}_{3,3}}(x_3'),\mu_{\bar{A}_{3,3}}($ $= [0.793,0.994$	$[\mu_{\underline{A}_{3,2}}(x_3'),\mu_{\bar{A}_{3,2}}$ $= [0.7,0.999]$	$[\mu_{\underline{A}_{3,4}}(x_3'),\mu_{\bar{A}_{3,4}}($ $= [0.75,0.999]$	$[\mu_{\underline{A}_{3,4}}(x_3'),\mu_{\bar{A}_{3,4}}$ $= [0.75,0.99]$	$[\mu_{\underline{A}_{3,1}}(x_3'),\mu_{\bar{A}_{3,1}}$ $= [0.75,0.99]$	$[\mu_{\underline{A}_{3,4}}(x_3'),\mu_{\bar{A}_{3,4}}$ $= [0.8,1]$	..	$[\mu_{\underline{A}_{3,4}}(x_3'),\mu_{\bar{A}_{3,4}}$ $= [0.75,0.99]$
x_4	$[\mu_{\underline{A}_{4,1}}(x_4'),\mu_{\bar{A}_{4,1}}($ $= [0.38,0.7]$	$[\mu_{\underline{A}_{4,1}}(x_4'),\mu_{\bar{A}_{4,1}}$ $= [0.54,0.82]$	$[\mu_{\underline{A}_{4,2}}(x_4'),\mu_{\bar{A}_{4,2}}($ $= [0.8,1]$	$[\mu_{\underline{A}_{4,1}}(x_4'),\mu_{\bar{A}_{4,1}}$ $= [0.46,0.76]$	$[\mu_{\underline{A}_{4,2}}(x_4'),\mu_{\bar{A}_{4,2}}$ $= [0.67,0.88]$	$[\mu_{\underline{A}_{4,2}}(x_4'),\mu_{\bar{A}_{4,2}}$ $= [0.8,1]$	..	$[\mu_{\underline{A}_{4,2}}(x_4'),\mu_{\bar{A}_{4,2}}$ $= [0.2,0.58]$
x_5	$[\mu_{\underline{A}_{5,3}}(x_5'),\mu_{\bar{A}_{5,3}}($ $= [0.7,0.94]$	$[\mu_{\underline{A}_{5,1}}(x_5'),\mu_{\bar{A}_{5,1}}$ $= [0.39,0.63]$	$[\mu_{\underline{A}_{5,2}}(x_5'),\mu_{\bar{A}_{5,2}}($ $= [0.067,0.5]$	$[\mu_{\underline{A}_{5,2}}(x_5'),\mu_{\bar{A}_{5,2}}$ $= [0.7,1]$	$[\mu_{\underline{A}_{5,2}}(x_5'),\mu_{\bar{A}_{5,2}}$ $= [0.067,0.5]$	$[\mu_{\underline{A}_{5,1}}(x_5'),\mu_{\bar{A}_{5,1}}$ $= [0.33,0.68]$	..	$[\mu_{\underline{A}_{5,1}}(x_5'),\mu_{\bar{A}_{5,1}}$ $= [0.43,0.67]$
x_6	$[\mu_{\underline{A}_{6,1}}(x_6'),\mu_{\bar{A}_{6,1}}($ $= [0.68,0.95]$	$[\mu_{\underline{A}_{6,1}}(x_6'),\mu_{\bar{A}_{6,1}}$ $= [0.68,0.95]$	$[\mu_{\underline{A}_{6,1}}(x_6'),\mu_{\bar{A}_{6,1}}($ $= [0.68,0.95]$	$[\mu_{\underline{A}_{6,2}}(x_6'),\mu_{\bar{A}_{6,2}}$ $= [0.8,1]$	$[\mu_{\underline{A}_{6,2}}(x_6'),\mu_{\bar{A}_{6,2}}$ $= [0.68,0.95]$	$[\mu_{\underline{A}_{6,1}}(x_6'),\mu_{\bar{A}_{6,1}}$ $= [0.8,1]$	..	$[\mu_{\underline{A}_{6,1}}(x_6'),\mu_{\bar{A}_{6,1}}$ $= [0.68,0.95]$
x_7	$[\mu_{\underline{A}_{7,3}}(x_7'),\mu_{\bar{A}_{7,3}}($ $= [0.7,0.923]$	$[\mu_{\underline{A}_{7,1}}(x_7'),\mu_{\bar{A}_{7,1}}$ $= [0.65,0.92]$	$[\mu_{\underline{A}_{7,3}}(x_7'),\mu_{\bar{A}_{7,3}}$ $= [0.7,0.923]$	$[\mu_{\underline{A}_{7,1}}(x_7'),\mu_{\bar{A}_{7,1}}$ $= [0.65,0.92]$	$[\mu_{\underline{A}_{7,3}}(x_7'),\mu_{\bar{A}_{7,3}}$ $= [0.7,0.921]$	$[\mu_{\underline{A}_{7,3}}(x_7'),\mu_{\bar{A}_{7,3}}$ $= [0.7,0.92]$	..	$[\mu_{\underline{A}_{7,1}}(x_7'),\mu_{\bar{A}_{7,1}}$ $= [0.65,0.92]$
x_8	$[\mu_{\underline{A}_{8,2}}(x_8'),\mu_{\bar{A}_{8,2}}($ $= [0.18,0.42]$	$[\mu_{\underline{A}_{8,3}}(x_8'),\mu_{\bar{A}_{8,3}}$ $= [0.17,0.42]$	$[\mu_{\underline{A}_{8,3}}(x_8'),\mu_{\bar{A}_{8,3}}$ $= [0.55,0.79]$	$[\mu_{\underline{A}_{8,2}}(x_8'),\mu_{\bar{A}_{8,2}}$ $= [0.53,0.74]$	$[\mu_{\underline{A}_{8,3}}(x_8'),\mu_{\bar{A}_{8,3}}$ $= [0.29,0.53]$	$[\mu_{\underline{A}_{8,2}}(x_8'),\mu_{\bar{A}_{8,2}}$ $= [0.22,0.47]$	..	$[\mu_{\underline{A}_{8,2}}(x_8'),\mu_{\bar{A}_{8,2}}$ $= [0.8,1]$
x_9	$[\mu_{\underline{A}_{9,1}}(x_9'),\mu_{\bar{A}_{9,1}}($ $= [0.7,0.94]$	$[\mu_{\underline{A}_{9,1}}(x_9'),\mu_{\bar{A}_{9,1}}$ $= [0.7,0.94]$	$[\mu_{\underline{A}_{9,2}}(x_9'),\mu_{\bar{A}_{9,2}}$ $= [0.7,0.97]$	$[\mu_{\underline{A}_{9,1}}(x_9'),\mu_{\bar{A}_{9,1}}$ $= [0.7,0.94]$	$[\mu_{\underline{A}_{9,1}}(x_9'),\mu_{\bar{A}_{9,1}}$ $= [0.7,0.94]$	$[\mu_{\underline{A}_{9,2}}(x_9'),\mu_{\bar{A}_{9,2}}$ $= [0.7,0.94]$	..	$[\mu_{\underline{A}_{9,2}}(x_9'),\mu_{\bar{A}_{9,2}}$ $= [0.7,0.97]$
x_{10}	$[\mu_{\underline{A}_{10,2}}(x_{10}'),\mu_{\bar{A}_1}$ $= [0.4,0.73]$	$[\mu_{\underline{A}_{10,1}}(x_{10}'),\mu_{\bar{A}}$ $= [0.75,1]$	$[\mu_{\underline{A}_{10,1}}(x_{10}'),\mu_{\bar{A}_1}$ $= [0.75,1]$	$[\mu_{\underline{A}_{10,1}}(x_{10}'),\mu_{\bar{A}}$ $= [0.75,1]$	$[\mu_{\underline{A}_{10,2}}(x_{10}'),\mu_{\bar{A}}$ $= [0.53,0.82]$	$[\mu_{\underline{A}_{10,2}}(x_{10}'),\mu_{\bar{A}}$ $= [40.2,0.59]$	..	$[\mu_{\underline{A}_{10,2}}(x_{10}'),\mu_{\bar{A}}$ $= [0.33,0.68]$
x_{11}	$[\mu_{\underline{A}_{11,2}}(x_{11}'),\mu_{\bar{A}_1}$ $= [0.7,1]$	$[\mu_{\underline{A}_{11,1}}(x_{11}'),\mu_{\bar{A}}$ $= [0.6,0.95]$	$[\mu_{\underline{A}_{11,1}}(x_{11}'),\mu_{\bar{A}_1}$ $= [0.6,0.95]$	$[\mu_{\underline{A}_{11,1}}(x_{11}'),\mu_{\bar{A}}$ $= [0.6,0.95]$	$[\mu_{\underline{A}_{11,2}}(x_{11}'),\mu_{\bar{A}}$ $= [0.7,1]$	$[\mu_{\underline{A}_{11,2}}(x_{11}'),\mu_{\bar{A}}$ $= [0.7,1]$	..	$[\mu_{\underline{A}_{11,2}}(x_{11}'),\mu_{\bar{A}}$ $= [0.7,1]$
x_{12}	$[\mu_{\underline{A}_{12,1}}(x_{12}'),\mu_{\bar{A}_1}$ $= [0.75,0.994]$	$[\mu_{\underline{A}_{12,1}}(x_{12}'),\mu_{\bar{A}}$ $= [0.75,0.994]$	$[\mu_{\underline{A}_{12,1}}(x_{12}'),\mu_{\bar{A}_1}$ $= [0.75,0.994]$	$[\mu_{\underline{A}_{12,4}}(x_{12}'),\mu_{\bar{A}}$ $= [0.5,0.88]$	$[\mu_{\underline{A}_{12,1}}(x_{12}'),\mu_{\bar{A}}$ $= [0.75,1]$	$[\mu_{\underline{A}_{12,3}}(x_{12}'),\mu_{\bar{A}}$ $= [0.75,0.999]$	..	$[\mu_{\underline{A}_{12,1}}(x_{12}'),\mu_{\bar{A}}$ $= [0.75,0.999]$
x_{13}	$[\mu_{\underline{A}_{13,3}}(x_{13}'),\mu_{\bar{A}_1}$ $= [0.75,1]$	$[\mu_{\underline{A}_{13,1}}(x_{13}'),\mu_{\bar{A}}$ $= [0.67,0.87]$	$[\mu_{\underline{A}_{13,2}}(x_{13}'),\mu_{\bar{A}_1}$ $= [0.6,0.8]$	$[\mu_{\underline{A}_{13,3}}(x_{13}'),\mu_{\bar{A}}$ $= [0.75,1]$	$[\mu_{\underline{A}_{13,2}}(x_{13}'),\mu_{\bar{A}}$ $= [0.75,0.994]$	$[\mu_{\underline{A}_{13,3}}(x_{13}'),\mu_{\bar{A}}$ $= [0.7,0.999]$	..	$[\mu_{\underline{A}_{13,3}}(x_{13}'),\mu_{\bar{A}}$ $= [0.75,1]$

For this an input vector x', we must repeat this procedure for all rules. The firing intervals of the rules are computed using the minimum function as the following:

$$\mathrm{R}^1\!:\ [\underline{a}^1,\overline{a}^1]=\begin{bmatrix}\min\{\mu_{\underline{A}_{1,3}}(x'_1),\mu_{\underline{A}_{2,1}}(x'_2),\mu_{\underline{A}_{3,3}}(x'_3),\mu_{\underline{A}_{4,1}}(x'_4),\mu_{\underline{A}_{5,3}}(x'_5),\mu_{\underline{A}_{6,1}}(x'_6),\mu_{\underline{A}_{7,3}}(x'_7),\mu_{\underline{A}_{8,2}}(x'_8),\mu_{\underline{A}_{9,1}}(x'_9),\mu_{\underline{A}_{10,2}}(x'_{10}),\mu_{\underline{A}_{11,2}}(x'_{11}),\mu_{\underline{A}_{12,1}}(x'_{12}),\mu_{\underline{A}_{13,3}}(x'_{13})\},\\ \min\{\mu_{\overline{A}_{1,3}}(x'_1),\mu_{\overline{A}_{2,1}}(x'_2),\mu_{\overline{A}_{3,3}}(x'_3),\mu_{\overline{A}_{4,1}}(x'_4),\mu_{\overline{A}_{5,3}}(x'_5),\mu_{\overline{A}_{6,1}}(x'_6),\mu_{\overline{A}_{7,3}}(x'_7),\mu_{\overline{A}_{8,2}}(x'_8),\mu_{\overline{A}_{9,1}}(x'_9),\mu_{\overline{A}_{10,2}}(x'_{10}),\mu_{\overline{A}_{11,2}}(x'_{11}),\mu_{\overline{A}_{12,1}}(x'_{12}),\mu_{\overline{A}_{13,3}}(x'_{13})\}\end{bmatrix}$$

$$\mathrm{R}^1\!:\ [\underline{a}^1,\overline{a}^1]=\begin{bmatrix}\min\{0.56,0.8,0.793,0.38,0.7,0.68,0.7,0.18,0.7,0.4,0.7,0.75,0.75\},\\ \min\{0.73,0.998,0.994,0.7,0.94,0.95,0.923,0.42,0.94,0.73,1,0.994,1\}\end{bmatrix}=[0.18,0.42]$$

$$R^2\!:\ [\underline{a}^2,\overline{a}^2]=\begin{bmatrix}\min\{\mu_{\underline{A}_{1,2}}(x'_1),\mu_{\underline{A}_{2,1}}(x'_2),\mu_{\underline{A}_{3,2}}(x'_3),\mu_{\underline{A}_{4,1}}(x'_4),\mu_{\underline{A}_{5,1}}(x'_5),\mu_{\underline{A}_{6,1}}(x'_6),\mu_{\underline{A}_{7,1}}(x'_7),\mu_{\underline{A}_{8,3}}(x'_8),\mu_{\underline{A}_{9,1}}(x'_9),\mu_{\underline{A}_{10,1}}(x'_{10}),\mu_{\underline{A}_{11,1}}(x'_{11}),\mu_{\underline{A}_{12,1}}(x'_{12}),\mu_{\underline{A}_{13,1}}(x'_{13})\},\\ \min\{\mu_{\overline{A}_{1,2}}(x'_1),\mu_{\overline{A}_{2,1}}(x'_2),\mu_{\overline{A}_{3,2}}(x'_3),\mu_{\overline{A}_{4,1}}(x'_4),\mu_{\overline{A}_{5,1}}(x'_5),\mu_{\overline{A}_{6,1}}(x'_6),\mu_{\overline{A}_{7,1}}(x'_7),\mu_{\overline{A}_{8,3}}(x'_8),\mu_{\overline{A}_{9,1}}(x'_9),\mu_{\overline{A}_{10,1}}(x'_{10}),\mu_{\overline{A}_{11,1}}(x'_{11}),\mu_{\overline{A}_{12,1}}(x'_{12}),\mu_{\overline{A}_{13,1}}(x'_{13})\}\end{bmatrix}$$

$$\mathrm{R}^2\!:\ [\underline{a}^2,\overline{a}^2]=\begin{bmatrix}\min\{0.6,0.98,0.7,0.54,0.39,0.68,0.65,0.17,0.7,0.75,0.6,0.75,0.67\},\\ \min\{0.86,1,0.999,0.82,0.63,0.95,0.92,0.42,0.94,1,0.95,0.999,0.87\}\end{bmatrix}=[0.17,0.42],$$

$$R^3\!:\ [\underline{a}^3,\overline{a}^3]=\begin{bmatrix}\min\{\mu_{\underline{A}_{1,2}}(x'_1),\mu_{\underline{A}_{2,2}}(x'_2),\mu_{\underline{A}_{3,4}}(x'_3),\mu_{\underline{A}_{4,2}}(x'_4),\mu_{\underline{A}_{5,2}}(x'_5),\mu_{\underline{A}_{6,1}}(x'_6),\mu_{\underline{A}_{7,3}}(x'_7),\mu_{\underline{A}_{8,3}}(x'_8),\mu_{\underline{A}_{9,2}}(x'_9),\mu_{\underline{A}_{10,1}}(x'_{10}),\mu_{\underline{A}_{11,1}}(x'_{11}),\mu_{\underline{A}_{12,1}}(x'_{12}),\mu_{\underline{A}_{13,2}}(x'_{13})\},\\ \min\{\mu_{\overline{A}_{1,2}}(x'_1),\mu_{\overline{A}_{2,2}}(x'_2),\mu_{\overline{A}_{3,4}}(x'_3),\mu_{\overline{A}_{4,2}}(x'_4),\mu_{\overline{A}_{5,2}}(x'_5),\mu_{\overline{A}_{6,1}}(x'_6),\mu_{\overline{A}_{7,3}}(x'_7),\mu_{\overline{A}_{8,3}}(x'_8),\mu_{\overline{A}_{9,2}}(x'_9),\mu_{\overline{A}_{10,1}}(x'_{10}),\mu_{\overline{A}_{11,1}}(x'_{11}),\mu_{\overline{A}_{12,1}}(x'_{12}),\mu_{\overline{A}_{13,2}}(x'_{13})\}\end{bmatrix}$$

$$\mathrm{R}^3\!:\ [\underline{a}^3,\overline{a}^3]=\begin{bmatrix}\min\{0.8,0.83,0.75,0.8,0.067,0.68,0.7,0.55,0.7,0.75,0.6,0.75,0.67\},\\ \min\{1,1,0.999,1,0.5,0.95,0.92,0.79,0.79,1,0.95,0.999,0.87\}\end{bmatrix}=[0.067,0.5]$$

$$R^4\!:\ [\underline{a}^4,\overline{a}^4]=\begin{bmatrix}\min\{\mu_{\underline{A}_{1,2}}(x'_1),\mu_{\underline{A}_{2,2}}(x'_2),\mu_{\underline{A}_{3,4}}(x'_3),\mu_{\underline{A}_{4,1}}(x'_4),\mu_{\underline{A}_{5,2}}(x'_5),\mu_{\underline{A}_{6,2}}(x'_6),\mu_{\underline{A}_{7,1}}(x'_7),\mu_{\underline{A}_{8,2}}(x'_8),\mu_{\underline{A}_{9,1}}(x'_9),\mu_{\underline{A}_{10,1}}(x'_{10}),\mu_{\underline{A}_{11,1}}(x'_{11}),\mu_{\underline{A}_{12,4}}(x'_{12}),\mu_{\underline{A}_{13,3}}(x'_{13})\},\\ \min\{\mu_{\overline{A}_{1,2}}(x'_1),\mu_{\overline{A}_{2,2}}(x'_2),\mu_{\overline{A}_{3,4}}(x'_3),\mu_{\overline{A}_{4,1}}(x'_4),\mu_{\overline{A}_{5,2}}(x'_5),\mu_{\overline{A}_{6,2}}(x'_6),\mu_{\overline{A}_{7,1}}(x'_7),\mu_{\overline{A}_{8,2}}(x'_8),\mu_{\overline{A}_{9,1}}(x'_9),\mu_{\overline{A}_{10,1}}(x'_{10}),\mu_{\overline{A}_{11,1}}(x'_{11}),\mu_{\overline{A}_{12,4}}(x'_{12}),\mu_{\overline{A}_{13,3}}(x'_{13})\}\end{bmatrix}$$

$$\mathrm{R}^4\!:\ [\underline{a}^4,\overline{a}^4]=\begin{bmatrix}\min\{0.8,0.83,0.75,0.46,0.7,0.8,0.65,0.53,0.7,0.75,0.6,0.5,0.75,\},\\ \min\{1,1,0.999,0.76,1,1,0.92,0.74,0.94,1,0.95,0.88,1\}\end{bmatrix}=[0.46,0.74]$$

$$R^5\!:\ [\underline{a}^5,\overline{a}^5]=\begin{bmatrix}\min\{\mu_{\underline{A}_{1,3}}(x'_1),\mu_{\underline{A}_{2,2}}(x'_2),\mu_{\underline{A}_{3,1}}(x'_3),\mu_{\underline{A}_{4,2}}(x'_4),\mu_{\underline{A}_{5,2}}(x'_5),\mu_{\underline{A}_{6,2}}(x'_6),\mu_{\underline{A}_{7,3}}(x'_7),\mu_{\underline{A}_{8,3}}(x'_8),\mu_{\underline{A}_{9,1}}(x'_9),\mu_{\underline{A}_{10,2}}(x'_{10}),\mu_{\underline{A}_{11,2}}(x'_{11}),\mu_{\underline{A}_{12,1}}(x'_{12}),\mu_{\underline{A}_{13,2}}(x'_{13})\},\\ \min\{\mu_{\overline{A}_{1,3}}(x'_1),\mu_{\overline{A}_{2,2}}(x'_2),\mu_{\overline{A}_{3,1}}(x'_3),\mu_{\overline{A}_{4,2}}(x'_4),\mu_{\overline{A}_{5,2}}(x'_5),\mu_{\overline{A}_{6,2}}(x'_6),\mu_{\overline{A}_{7,3}}(x'_7),\mu_{\overline{A}_{8,3}}(x'_8),\mu_{\overline{A}_{9,1}}(x'_9),\mu_{\overline{A}_{10,2}}(x'_{10}),\mu_{\overline{A}_{11,2}}(x'_{11}),\mu_{\overline{A}_{12,1}}(x'_{12}),\mu_{\overline{A}_{13,2}}(x'_{13})\}\end{bmatrix}$$

$$\mathrm{R}^5\!:\ [\underline{a}^5,\overline{a}^5]=\begin{bmatrix}\min\{0.13,0.98,0.75,0.67,0.067,0.68,0.7,0.29,0.7,0.53,0.7,0.75,0.75\},\\ \min\{0.55,1,0.999,0.88,0.5,0.95,0.92,0.53,0.94,0.82,1,1,1\}\end{bmatrix}=[0.067,0.5]$$

$$R^6\!:\ [\underline{a}^6,\overline{a}^6]=\begin{bmatrix}\min\{\mu_{\underline{A}_{1,3}}(x'_1),\mu_{\underline{A}_{2,2}}(x'_2),\mu_{\underline{A}_{3,1}}(x'_3),\mu_{\underline{A}_{4,2}}(x'_4),\mu_{\underline{A}_{5,2}}(x'_5),\mu_{\underline{A}_{6,2}}(x'_6),\mu_{\underline{A}_{7,3}}(x'_7),\mu_{\underline{A}_{8,3}}(x'_8),\mu_{\underline{A}_{9,1}}(x'_9),\mu_{\underline{A}_{10,2}}(x'_{10}),\mu_{\underline{A}_{11,2}}(x'_{11}),\mu_{\underline{A}_{12,1}}(x'_{12}),\mu_{\underline{A}_{13,2}}(x'_{13})\},\\ \min\{\mu_{\overline{A}_{1,3}}(x'_1),\mu_{\overline{A}_{2,2}}(x'_2),\mu_{\overline{A}_{3,1}}(x'_3),\mu_{\overline{A}_{4,2}}(x'_4),\mu_{\overline{A}_{5,2}}(x'_5),\mu_{\overline{A}_{6,2}}(x'_6),\mu_{\overline{A}_{7,3}}(x'_7),\mu_{\overline{A}_{8,3}}(x'_8),\mu_{\overline{A}_{9,1}}(x'_9),\mu_{\overline{A}_{10,2}}(x'_{10}),\mu_{\overline{A}_{11,2}}(x'_{11}),\mu_{\overline{A}_{12,1}}(x'_{12}),\mu_{\overline{A}_{13,2}}(x'_{13})\}\end{bmatrix}$$

$$\mathrm{R}^6\!:\ [\underline{a}^6,\overline{a}^6]=\begin{bmatrix}\min\{0.47,0.83,0.8,0.8,0.33,0.8,0.7,0.22,0.7,0.2,0.7,0.75,0.7\},\\ \min\{063,1,1,1,0.68,1,0.92,0.47,0.94,0.59,1,0.999,0.994\}\end{bmatrix}=[0.2,0.47]$$

$$R^n\!:\ [\underline{a}^n,\overline{a}^n]=\begin{bmatrix}\min\{\mu_{\underline{A}_{1,1}}(x'_1),\mu_{\underline{A}_{2,2}}(x'_2),\mu_{\underline{A}_{3,4}}(x'_3),\mu_{\underline{A}_{4,2}}(x'_4),\mu_{\underline{A}_{5,1}}(x'_5),\mu_{\underline{A}_{6,1}}(x'_6),\mu_{\underline{A}_{7,1}}(x'_7),\mu_{\underline{A}_{8,2}}(x'_8),\mu_{\underline{A}_{9,2}}(x'_9),\mu_{\underline{A}_{10,2}}(x'_{10}),\mu_{\underline{A}_{11,2}}(x'_{11}),\mu_{\underline{A}_{12,1}}(x'_{12}),\mu_{\underline{A}_{13,3}}(x'_{13})\},\\ \min\{\mu_{\overline{A}_{1,1}}(x'_1),\mu_{\overline{A}_{2,2}}(x'_2),\mu_{\overline{A}_{3,4}}(x'_3),\mu_{\overline{A}_{4,2}}(x'_4),\mu_{\overline{A}_{5,1}}(x'_5),\mu_{\overline{A}_{6,1}}(x'_6),\mu_{\overline{A}_{7,1}}(x'_7),\mu_{\overline{A}_{8,2}}(x'_8),\mu_{\overline{A}_{9,2}}(x'_9),\mu_{\overline{A}_{10,2}}(x'_{10}),\mu_{\overline{A}_{11,2}}(x'_{11}),\mu_{\overline{A}_{12,1}}(x'_{12}),\mu_{\overline{A}_{13,3}}(x'_{13})\}\end{bmatrix}$$

$$\mathrm{R}^n\!:\ [\underline{a}^n,\overline{a}^n]=\begin{bmatrix}\min\{0.5,0.83,0.75,0.2,0.43,0.68,0.65,0.8,0.7,033,0.7,0.75,0.75\},\\ \min\{0.67,1,0.999,0.58,0.67,0.95,0.92,1,0.97,0.68,1,0.999,1\}\end{bmatrix}=[0.2,0.58]$$

We have to complete these procedures for all rules. Consequently, we obtain the Table 2.

Table 2. The firing intervals and theirs consequents for rules

No. of rule	Firing Interval	Consequent
R^1	$[\underline{a}^1,\overline{a}^1]=[0.18,0.42]$	$[\underline{y}^1,\overline{y}^1]=[0.146,0.228]$
R^2	$[\underline{a}^2,\overline{a}^2]=[0.17,0.42]$	$[\underline{y}^2,\overline{y}^2]=[0.149,0.23]$
R^3	$[\underline{a}^3,\overline{a}^3]=[0.067,0.5]$	$[\underline{y}^3,\overline{y}^3]=[0.18,0.19]$
R^4	$[\underline{a}^4,\overline{a}^4]=[0.46,0.74]$	$[\underline{y}^4,\overline{y}^4]=[0.162,0.174]$
R^5	$[\underline{a}^5,\overline{a}^5]=[0.067,0.5]$	$[\underline{y}^5,\overline{y}^5]=[0.91,0.92]$
R^6	$[\underline{a}^6,\overline{a}^6]=[0.2,0.47]$	$[\underline{y}^6,\overline{y}^6]=[0.896,0.96]$
:	:	:
R^n	$[\underline{a}^n,\overline{a}^n]=[0.2,0.58]$	$[\underline{y}^n,\overline{y}^n]=[0.949,0.96]$

The TR for an IT2 FLS uses the *iterative* KM algorithms, this may cause a computational bottleneck, even for real medical applications of the IT2 FLSs, where it has been demonstrated that very good performance can be satisfied by doing this. From (75) and (76), we are calculated left and right endpoints $[\,y_l,y_r]$ as follows:

$$y_l=\frac{\overline{a}^1\underline{y}^1+\underline{a}^2\underline{y}^2+\underline{a}^3\underline{y}^3+\underline{a}^4\underline{y}^4+\underline{a}^5\underline{y}^5+\underline{a}^6\underline{y}^6+\cdots+\underline{a}^n\underline{y}^n}{\overline{a}^1+\underline{a}^2+\underline{a}^3+\underline{a}^4+\underline{a}^5+\underline{a}^6+\cdots+\underline{a}^n},$$

$$y_l=\frac{0.42(0.146)+0.17(0.149)+0.067(0.18)+0.46(0.162)+0.067(0.91)+0.2(0.896)+\cdots+0.2(0.949)}{0.42+0.17+0.067+0.46+0.067+0.2+\cdots+0.2},$$

$$y_r = \frac{\underline{a}^1\overline{y}^1 + \underline{a}^2\overline{y}^2 + \underline{a}^3\overline{y}^3 + \underline{a}^4\overline{y}^4 + \underline{a}^5\overline{y}^5 + \underline{a}^6\overline{y}^6 + \cdots + \overline{a}^n\overline{y}^n}{\underline{a}^1 + \underline{a}^2 + \underline{a}^3 + \underline{a}^4 + \underline{a}^5 + \underline{a}^6 + \cdots + \overline{a}^n},$$

$$y_r = \frac{0.18(0.228) + 0.17(0.23) + 0.067(0.19) + 0.46(0.174) + 0.067(0.92) + 0.2(0.96) + \cdots + 0.58(0.96)}{0.18 + 0.17 + 0.067 + 0.46 + 0.067 + 0.2 + \cdots + 0.58}.$$

Therefore, we obtained the values of left and right endpoints y_l and y_r, respectively. Finally, we are defuzzified the interval set (output of the IT2 FLS) in order to compute the crisp output y of the IT2 FLS using (82). We repeated all these procedure with each an input vectors (all observations) to compute the crisp output y_i $(i = 1, \ldots, 270)$ of the IT2 FLS for all cases.

6.2. Software Performance

First part the performance of *Matlab* is used a function "*IT2FLS*" that is provided for computing the output of an IT2 FLS. For performance, the program of "*IT2FLS*" we need to given the rule-base and inputs for the problem. The above application is performed using the function "*IT2FLS*". A nine-point vector $[p^1, p^2, \ldots, p^9]$ represents each IT2 FS. Therefore, the IT2 FS $\widehat{A}_{1,1}$ is represented as (0.3 0.3 0.35 0.65 0.3 0.3 0.36 0.6 0.8), $\widehat{A}_{1,2}$ is represented as (0.4 0.62 0.68 0.9 0.5 0.62 0.68 0.8 0.8), and $\widehat{A}_{1,3}$ is represented as (0.65 0.95 1 1 0.7 0.94 1 1 0.8), such shown in Fig. 7.

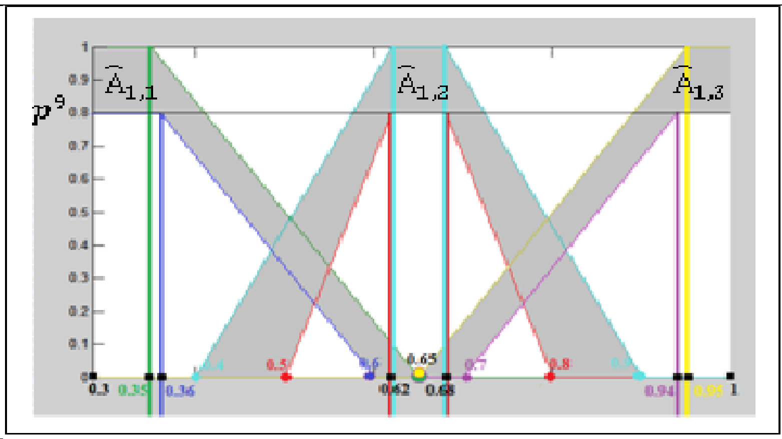
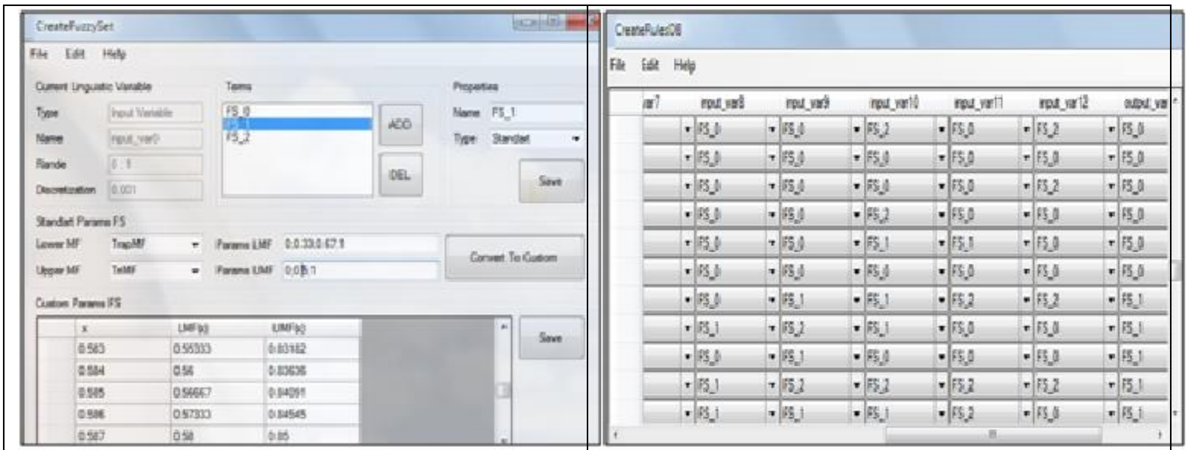


Fig. 7. The MFs $\hat{A}_{1,1}, \hat{A}_{1,2}$, and , $\hat{A}_{1,3}$ of the IT2 FLS for the first input



(A) (B)

Fig. 8. (A) Creation fuzzy set in IT2 FLS. (B) Rule-base in IT2 FLS of heart diseases application in C#

Second part of performance is performed using IT2FLS software in visual C# that includes some modules as the linguistic variable, the mf, the rule-base, and the simulator editors. The linguistic variable is used to define the input and the output linguistic variables. The MF is used to define the membership functions related with the linguistic variable. IT2FLS software also allows user to creating and editing rules. The simulator is used to present an interactive view of the logic inference. IT2FLS software does not limit the number of linguistic variables, membership functions of linguistic variables and rules. For our application, we have created the MFs and the rules that are depicted in Figure 8.

6.3. Experimental Comparison

In this subsection, we compare the performance of the four methods. We present results of a comparison of *Heart Diseases (HDs)* using an intelligent architecture between interval type-2 fuzzy logic systems using the IT2FLS in MATLAB and the IT2FLS in Visual C# models with type-1 fuzzy inference systems (Mamdani, and Takagi-Sugeno). The prediction root mean square error (RMSE) was 0.00075. Table 3 shows the various RMSE of four predicting methods, where the IT2 FLS in MATLAB and the IT2FLS in Visual C# evaluate the best *HDs* predicts respectively. The advantage of using the IT2FLS predicting method is that it obtains better results, even when data contains high uncertainty. The IT2 provide rather modest performance improvements over the T1 predictor.

The comparison of the predictions RMSE for three methods is shown in Fig. 9. Observe from Fig. 9 that: the blue line represents the prediction errors between the actual output of HDs and prediction output using T1 FLS, that has limitation between -0.5 and 0.5; The green line represents the prediction errors using IT2 FLS in C# limited by [-0.1435,0.2325]. While, the prediction errors using IT2 FLS in Matlab has limited by [-0.2194, 0.1385], that are represented by the red line. Note that, the best prediction RMSE when used IT2 FLS in Matlab that was closer to zero.

Table 3. The various RMSEs of four predicting methods

Method of model	RMSE
IT2 FLS in MATLAB	0.00075
IT2FLS in visual C#	0.0253
T1 FIS (Mamdani)	0.2441
T1 FIS (Takagi-Sugeno)	0.1988

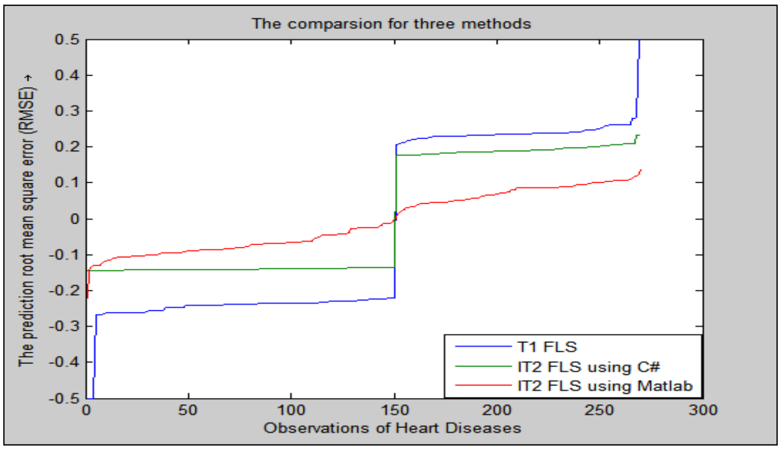


Fig. 9. The comparison of the predictions RMSE for three methods

7. CONCLUSIONS

This work shows that expression (11), which was given by Mendel et al. but on discrete domain, in order to prove an IT2 FS $\hat{A}$ is the union of countable-infinity number of embedded IT2 FSs for a continuous IT2. We extended the derivation of the *union* of *two* IT2 FSs, which was given by Mendel and Bob John, to the *intersection* and *union* of *N* IT2 FSs, depending on the various concepts such theorem 2 and 3. We presented the *meet* operation of *N* IT2 FSs depending on the concept of the secondary MF such theorem 4, which was given by Karnik and Mendel but for the *join* operation. Theorem 5a and 5b provided the derivation of the relationship between the consequent and the DOU of the T2 fired output for SF and NSF. We have provided the derivation of the general form for continuous domain to calculate the different kinds of type-reduced, which was given by Karnik et al. but for discrete domain. Additionally, We applied the medical application of IT2 FLS's to HDs, in which it demonstrated the basic ideas and the mathematical operations of IT2 fuzzy sets and systems. Finally, we have compared the performance of the four methods of HDs between IT2 FLS using the IT2FLS in MATLAB and the IT2FLS in Visual C# models with T1 FISs (Mamdani, and Takagi-Sugeno). The best result of RMSE was 0.00075 with the IT2FLS in MATLAB. The prediction errors using

IT2 FLS in Matlab has limited by [-0.2194, 0.1385], that are represented by the red line that was closer to zero. Our future work includes optimizing the knowledge base of the IT2 FLS, and modeling the IT2 FLS to neural network model on continuous domain.

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