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Observations Concerning Chordal Graph Polynomials

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Abstract: For every graph that is clique equivalent to a connected chordal graph, it is shown that the associated dependence polynomial has a unit root and that the associated clique and independence polynomials have negative unit roots. The dependence polynomial for a graph that is the join of two graphs is also shown to have a unit root when at least one of the two joined graphs is clique equivalent to a connected chordal graph. A condition satisfied by the eigenvalues of graphs that are clique equivalent to connected chordal graphs with clique numbers less than four is identified.

Keywords: chordal graph; graph polynomial; dependence polynomial; clique polynomial; independence polynomial; graph eigenvalues

1. Introduction

Because of their intrinsic graph theoretic properties and their utility when applied to certain aspects of biology, genetics, scheduling, psychology, database design, and computer science, chordal graphs have been studied extensively during the last several decades, e.g., [1, 2]. The dependence polynomial of a graph has also been the focus of attention in recent years, e.g., [3], in large part due to its relationship to the word problem for partially Abelian monoids [4]. Clique polynomials and independence polynomials of graphs have also been studied and their roots applied to providing bounds for certain graph invariants, as well as used to generate special fractal sets associated with graphs, e.g., [5-8].

This short paper shows that the dependence polynomial for graphs that are clique equivalent to connected chordal graphs – or for graphs that are the join of two graphs, at least one of which is clique equivalent to a connected chordal graph - has a unit root. It necessarily follows that clique polynomials for graphs that are clique equivalent to connected chordal graphs and independence polynomials for complements of graphs clique equivalent to connected chordal graphs have a negative unit root. This unit root property is applied to provide a condition that must be satisfied by the eigenvalues of graphs that are clique equivalent to connected chordal graphs with clique numbers less than four.

The remainder of this paper is organized as follows: relevant definitions and terminology are introduced in Section 2. Preliminary lemmas are provided in Section 3 and the main results are established in Section 4. Illustrative examples are presented in Section 5. Closing remarks comprise the final section of this paper.

2. Definitions and Terminology

A graph G is a pair $\big(V(G), E(G)\big)$, where $V(G)$ is a finite non-empty set of *vertices* and $E(G)$ is either a set of doubleton subsets of $V(G)$ called *edges* or the empty set. The *order* of G is the cardinality $|V(G)|$ of $V(G)$ and the *size* of G is the cardinality $|E(G)|$ of $E(G)$. The complement $\bar{G}$ of G is the pair $(V(\bar{G}), E(\bar{G}))$ with $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{\{u,v\} \subset V(G) : \{u,v\} \notin E(G)\}$. The *join* of graphs G_1 and G_2 is the graph $G_1 + G_2$ with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \big\{\{u,v\} | u \in V(G_1) \text{ and } v \in V(G_2)\big\}.$$

Two vertices $u, v \in V(G)$ are *adjacent* when $e = \{u,v\} \in E(G)$ in which case e is said to connect u and v. A $u - v$ *walk* is an alternating sequence of vertices and edges beginning with u and ending with v such that every edge connects the vertices immediately preceding and following it. A $u - v$ *path* is a $u - v$ walk in which no vertex is repeated. In this case u is said to be connected to v. G is *connected* if its order is one or if every two vertices in G are connected. A $u - v$ path for which $u = v$ and which contains at least three edges is a *cycle*. The *length* of a cycle is the number of edges contained within it and a *chord* of a cycle is an edge between

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two nonconsecutive vertices in the cycle. The number of 3-cycles (i.e., triangles) in G is $|T(G)|$.

G is a *chordal graph* if every cycle of length at least four has a chord. A *tree* is a connected chordal graph which has no cycles. A *complete graph* is a connected chordal graph in which every two of its vertices are adjacent. A graph F is a *subgraph* of G if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. A *clique* in G is either a vertex or a complete subgraph of G and is *maximal* if it is not a proper subgraph of another clique. The order of a clique is the cardinality of its vertex set and the *clique number* $\omega(G)$ of G is the maximum order among the maximal cliques of G. Let c_n be the number of cliques of order n in G and $\mathcal{N}_G \equiv (c_1, c_2, \cdots, c_{\omega(G)})$ be the ordered $\omega(G)$-tuple of the numbers of cliques of each order in G. Graphs G and $G^{'}$ are *clique equivalent* graphs (denoted $G \sim G^{'}$) when $\mathcal{N}_G = \mathcal{N}_{G^{'}}$. Clearly, $G \sim G$.

An *independent set* of vertices in G is a nonempty subset of $V(G)$ whose elements are pairwise non-adjacent. The maximum cardinality among the independent sets of G is the *independence number* $\alpha(G)$ of G. Let s_n be the number of *independent* sets of cardinality n in G and $\mathcal{J}_G = (s_1, s_2, \cdots, s_{\alpha(G)})$ be the ordered $\alpha(G)$-tuple of the numbers of independent sets of each order in G. Observe that c_n is a clique in G if and only if s_n is an independent set in $\bar{G}$ so that $\mathcal{N}_G = \mathcal{J}_{\bar{G}}$.

The *dependence polynomial* [3] of a graph G is

$$f_G(x) = 1 - \sum_{n=1}^{\omega(G)} (-1)^{n-1} c_n x^n, \qquad (*)$$

the *clique polynomial* [5] of G is

$$C_G(x) = 1 + \sum_{n=1}^{\omega(G)} c_n x^n, \qquad (\aleph)$$

and the *independence polynomial* [7] of G is

$$I_G(x) = 1 + \sum_{n=1}^{\alpha(G)} s_n x^n. \qquad (\beth)$$

Recall from Descartes' Rule of Signs that - because there are no sign changes in $f_G(-x)$ - all real roots of $f_G(x)$ are positive, whereas – since there are no sign changes in $C_G(x)$ and $I_G(x)$ – all real roots for these polynomials are negative.

3. Preliminary Lemmas

The following five lemmas are well known (e.g., [4]) and are used below. They are stated here as lemmas for the reader's convenience.

Lemma 1. $f_G(x) = C_G(-x) = I_{\bar{G}}(-x)$.

Lemma 2. *If G is a tree of order m, then* $f_G(x) = 1 - mx + (m-1)x^2$.

Lemma 3. *G is a complete graph of order m if and only if* $f_G(x) = (1-x)^m$.

Lemma 4. *If $G = G_1 + G_2$, then* $f_{G_1+G_2}(x) = f_{G_1}(x) f_{G_2}(x)$.

Lemma 5 [9]**.** *If G is a connected chordal graph, then*

$$\sum_{n=1}^{\omega(G)} (-1)^{n-1} c_n = 1. \qquad (\partial)$$

The next two results are required for proofs developed in the next section.

Lemma 6. *$G \sim G^{'}$ if and only if* $f_G(x) = f_{G^{'}}(x)$ $[\, C_G(x) = C_{G^{'}}(x)]$ $\{ I_{\bar{G}}(x) = I_{\overline{G^{'}}}(x)\}$.

Proof. $G \sim G^{'} \Leftrightarrow \mathcal{N}_G = \mathcal{N}_{G^{'}} \Leftrightarrow f_G(x) = f_{G^{'}}(x)$ $[C_G(x) = C_{G^{'}}(x)]$$\{\mathcal{J}_{\bar{G}} = \mathcal{J}_{\overline{G^{'}}} \Leftrightarrow I_{\bar{G}}(x) = I_{\overline{G^{'}}}\}$. ■

Lemma 7. *If $G \sim G^{'}$ and G is a connected chordal graph, then (∂) holds for $G^{'}$.*

Proof. This is an obvious consequence of the fact that $\mathcal{N}_{G^{'}} = \mathcal{N}_G$. ■

4. Main Results

Theorem 8. *If G is clique equivalent to a connected chordal graph, then $x = 1$ is a root of $f_G(x)$.*

Proof. From (*) it is always the case that $f_G(1) = 1 - \sum_{n=1}^{\omega(G)} (-1)^{n-1} c_n$. Since G is clique equivalent to a connected chordal graph, then it follows from lemmas 6 and 7 that $f_G(1) = 0$. ■

Corollary 9. $f_G(x) = (x-1)Q_G(x)$.

Proof. Since $x = 1$ is a root of $f_G(x)$, then $f_G(x) = (x-1)Q_G(x)$. ■

Corollary 10. $Q_G(0) = -1$.

Proof. $f_G(x) = (x-1)Q_G(x) \Rightarrow f_G(0) = -Q_G(0)$. However, from (*) it is always the case that $f_G(0) = 1$. Thus, $Q_G(0) = -1$. ■

Theorem 11. *If either of G_1 or G_2 is clique equivalent to a connected chordal graph, then $x = 1$ is a root of $f_{G_1+G_2}(x)$.*

Proof. From lemma 4 it follows that $f_{G_1+G_2}(1) = f_{G_1}(1) f_{G_2}(1)$. If $G_i, i \in \{1,2\}$, is clique equivalent to a connected chordal graph, then – from Theorem 8 - $f_{G_i}(1) = 0$ so that $f_{G_1+G_2}(1) = 0$. ■

Theorem 12. *Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of a graph G of order n and $\omega(G) < 4$. If G is clique equivalent to a connected chordal graph, then*

$$\sum_{i=1}^{n} \lambda_i^2 (3 - \lambda_i) = 6(n-1). \qquad (\spadesuit)$$

Proof. When $\omega(G) < 4$, then

$$f_G(x) = 1 - nx + |E(G)|x^2 - |T(G)|x^3 = 1 - nx + \left(\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2\right)x^2 - \left(\frac{1}{6}\sum_{i=1}^{n}\lambda_i^3\right)x^3,$$

where use has been made of the fact that $|E(G)| = \frac{1}{2}\sum_{i=1}^{n}\lambda_i^2$ and $|T(G)| = \frac{1}{6}\sum_{i=1}^{n}\lambda_i^3$. Since G is clique equivalent to a connected chordal graph, then – from lemma 7 - $f_G(1) = 0$ so that

$$1 - n + \left(\frac{1}{2}\sum_{i=1}^{n}\lambda_i^2\right) - \left(\frac{1}{6}\sum_{i=1}^{n}\lambda_i^3\right) = 0.$$

Upon rearrangement and after factoring, this yields the required result. ∎

Theorem 13. *If G is clique equivalent to a connected chordal graph, then $x = -1$ is a root of $C_G(x)$ and $I_{\bar{G}}(x)$.*
Proof. If G is clique equivalent to a connected chordal graph, then it follows from lemmas 1 and 6, that $C_G(-1) = I_{\bar{G}}(-1) = f_G(1) = 0$. ∎
Corollary 14. *$C_G(x) = (x+1)P_G(x)$ and $I_{\bar{G}}(x) = (x+1)R_{\bar{G}}(x)$.*
Proof. This is an obvious consequence of the fact that $x = -1$ is a root of $C_G(x)$ and a root of $I_{\bar{G}}(x)$. ∎
Corollary 15. $P_G(0) = 1 = R_{\bar{G}}(0)$.
Proof. From (א) and (ב) it is always the case that $C_G(0) = 1 = I_{\bar{G}}(0)$. If G is clique equivalent to a connected chordal graph, then $C_G(0) = P_G(0)$ and $I_{\bar{G}}(0) = R_{\bar{G}}(0)$ so that $P_G(0) = 1 = R_{\bar{G}}(0)$. ∎

5. Examples

- As examples which validate theorem 8 and corollaries 9 and 10, consider a tree G of order m and a complete graph H of order m. From lemmas 2 and 3 it is easily seen that $f_G(1) = 0 = f_H(1)$ (thereby validating theorem 8); and (*ii*) $Q_G(0) = -1$, where $Q_G(x) = \frac{1-mx+(m-1)x^2}{x-1} = (m-1)x - 1$ and $Q_H(0) = -1$, where $Q_H(x) = \frac{(1-x)^m}{(x-1)} = -(1-x)^{m-1}$ (thereby validating corollaries 9 and 10).

- If G is a complete graph of order 3, then its eigenvalues are -1 with multiplicity 2 and 2 with multiplicity 1. Substituting this into (♠) yields

  $2(-1)^2(3+1) + (2)^2(3-2) = 6(3-1)$
  or
  $12 = 12$
  (thereby validating theorem 12).

- Consider the connected chordal graph G and its complement $\bar{G}$ shown in Figure 1. By inspection it is determined that

$f_G(x) = 1 - 4x + 5x^2 - 2x^3$
and
$C_G(x) = 1 + 4x + 5x^2 + 2x^3 = I_{\bar{G}}(x)$
from which it is readily seen that $f_G(1) = 0 = C_G(-1) = I_{\bar{G}}(-1)$. The eigenvalues of G are

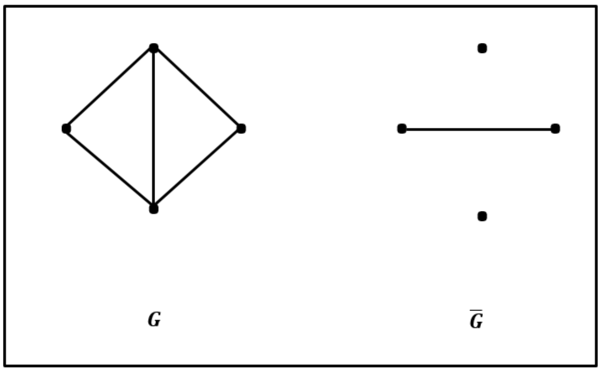

Figure 1. A connected chordal graph G and its complement $\bar{G}$.

$\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = \frac{1}{2}(1+\sqrt{17})$, and $\lambda_4 = \frac{1}{2}(1-\sqrt{17})$.
Substituting these values into (♠) verifies theorem 12, i.e.

$$0^2(3-0) + (-1)^2(3-(-1)) + \left[\tfrac{1}{2}(1+\sqrt{17})\right]^2\left[3 - \tfrac{1}{2}(1+\sqrt{17})\right] + \left[\tfrac{1}{2}(1-\sqrt{17})\right]^2\left[3 - \tfrac{1}{2}(1-\sqrt{17})\right] = 6(4-1)$$

or
$18 = 18$.

6. Closing Remarks
The primary objective of this paper has been to show that the dependence polynomials of graphs that are clique equivalent to connected chordal graphs have $x = 1$ as a root. This result was used to provide a condition that must be satisfied by the eigenvalues of all graphs with clique numbers less than four that are clique equivalent to connected chordal graphs. Accordingly, the contrapositive version of this condition serves to identify graphs with clique numbers less than four which are not clique equivalent to connected chordal graphs.

As ancillary results, it was also shown that: (*i*) the dependence polynomial for a graph that is the join of two graphs has a unit root when at least one of the two graphs is clique equivalent to a connected chordal graph; and (*ii*) the clique polynomials for graphs clique equivalent to connected chordal graphs and the independence polynomials of their graph complements have $x = -1$ as a root.

It is interesting to note that when applied to trees for which $\omega(G) = 2$, the cubic term in the eigenvalue condition of theorem 12 vanishes and (♠) becomes

$$\sum_{i=1}^{n} \lambda_i^2 = 2(n-1).$$

Observe that - when divided by 2 - this expression simply states the well-known fact that the size of a tree is one less than its order. Furthermore, for graphs with zero size, the left hand side of (♠) vanishes yielding

$$n = 1\,.$$

This is consistent with the fact that the only connected chordal graph with $\omega(G) = 1$ is the graph of order one and size zero.

As a final observation, it is noted that if G is an order n graph with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and $\omega(G) = 4$ that is also clique equivalent to a connected chordal graph, then an obvious consequence of Theorem 8 is the fact that the number c_4 of cliques of order 4 in G is given by

$$c_4 = \frac{1}{6}\sum_{i=1}^{n} \lambda_i^3 - \frac{1}{2}\sum_{i=1}^{n} \lambda_i^2 + n - 1.$$

As a simple illustration of this consider the complete graph of order 4. Since the associated eigenvalues are 3 of multiplicity 1 and -1 of multiplicity 3, then application of the last expression readily yields $c_4 = 1$ as the required result.

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