

# Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix $(XX)$ are Skewed

C. A. Uzuke<sup>1</sup> , J. I. Mbegbu<sup>2</sup>

<sup>1</sup>Department of Statistics, Nnamdi Azikiwe University Awka

<sup>2</sup>Department of Mathematics, University of Benin, Benin City Edo state Nigeria

**Abstract:** Methods of estimating the ridge parameter  $k$  in ridge regression analysis are available in the literature. This paper proposed some methods based on the works of Lawless and Wang (1976) and Khalaf and Shurkur (2005). A simulation study was conducted and mean square error (MSE) criterion was used to compare the performances of the proposed estimators and some other existing ones. It was observed that the performance of the these estimators depend on the variance of the random error  $\sigma$ , the correlation among the explanatory variables  $\rho$ , the sample size  $n$  and the number of explanatory variables  $p$ . The increase in the number of explanatory variables and increase in the sample size reduces the MSE of the estimators even when the correlation between the explanatory variables are high, but for small sample size, MSE increases as the values of  $p$  increases. One of the proposed methods  $k_{15}$  outperforms all the other existing and proposed methods considered in terms of MSE values.

**Keywords:** Eigenvalues; Mean Square Error; Multicollinearity; Ridge regression; Skewness.

### 1.0 Introduction

Ridge regression first introduced by Hoerl and Kennard (1970) is one of the most popular methods that have been suggested for solving the multicollinearity problem. In case of multicollinearity, ordinary least square (OLS) method produces unbiased estimates that could be very unstable due to their large variances which lead to poor prediction. Ridge Regression allows biased estimators of the ridge regression coefficients by modifying the least square method to attain substantial reduction in variance with an accompanied increase in stability of these coefficients. At this stage, the main aim is in finding the value of the ridge regression parameter  $k$  that reduces the variance but increases bias of the error term.

Hoerl and Kennard (1970) proved that there is a non zero value of such ridge parameter for which the mean square error (MSE) is smaller than the variance of the OLS estimator of the respective parameter

Many authors has worked in this area of research, developed and proposed different estimator for the ridge regression parameter. And some has applied these methods to a real life problems. Some of them

are: Hoerl and Kennard (1970), Hoerl et al (1975), Mc Donald and Galarneau (1975), Lawless and Wang (1976), Sale and Kibria (1996), Kibria (2003), Khalaf and Shurkur (2005), Zang and Ibrahim (2005), Alkhamisi et al (2006), Bata et al (2008), Alkassab and Qwaider (2010) Khalaf (2012 a,b) Cule and De lorio (2013) and Dorugade (2014). Lawless and Wang introduced the eigenvalues of the design matrix  $(XX)$  in the method of estimating  $k$  the ridge parameter and Khalaf and Shurkur (2005) used the maximum eigenvalues of the design matrix  $(XX)$  and obtained an improved ridge regression estimator that exhibited smaller MSE. In Kibria (2003) and Alkhamisi et al (2006), the authors used simulation technique to study the properties of the proposed estimators by comparing them with the existing estimators. They obtained MSEs that are smaller than those of the existing estimators and that of OLS.

Khalaf and Shurkur (2005) used the maximum eigenvalue of the matrix  $(XX)$  without taking into consideration the skewness of these eigenvalues. In this paper, we extended the work of Khalaf and Shurkur (2005) by proposing 6 more ridge estimators that will take care of the skewness of the eigenvalues

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C. A. Uzuke (Correspondence)

chinwe\_uzuke@yahoo.com

of the matrix  $(X'X)$  and also the model will be extended to 9 explanatory variables. The performances of these ridge estimators will be evaluated using MSE plotted on a graph via Monte Carlo simulation study.

The paper is organised as follows: In section 2, the ridge regression model was presented, in section 3, the design of the Monte Carlo simulation together with the factors that can affect the proposed ridge parameters are introduced. In section 4, we describe

$$Y = X\beta + \varepsilon$$

$$\text{where } X = \begin{pmatrix} x_{11} & \dots & \dots & x_{1p} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x_{n1} & \dots & \dots & x_{np} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix} \text{ and } Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The matrix  $X$  contains the values of  $p$  predictor variables at each of  $n$  observations,  $Y$  is a vector of the observed values,  $\beta$  is a vector of unknown coefficients and  $\varepsilon$  is a vector of experimental error with the properties  $E(\varepsilon) = 0$  and  $E(\varepsilon'\varepsilon) = \sigma^2 I_n$ , where  $I_n$  is an identity matrix of order  $n$ . For convenience,  $X$  and  $Y$  are centered and scaled so that  $(X'X)$  has the form of a correlation matrix.

The ordinary least square (OLS) estimate of  $\beta$  is given by;

$$\hat{\beta} = (X'X)^{-1} X' Y \quad (2.2)$$

The estimate  $\hat{\beta}$  is chosen to minimize the sum of square residuals;

$$SSR(\beta) = (Y - X\hat{\beta})'(Y - X\hat{\beta}) \quad (2.3)$$

The properties of  $\hat{\beta}$  are that (i) it is unbiased,  $E(\hat{\beta}) = \beta$  (ii) it has a minimum variance among all linear unbiased estimators;  $Var(\hat{\beta}) = \sigma^2 (X'X)^{-1}$ , which depends on the characteristics of the matrix,  $(X'X)$ . If this matrix is ill-conditioned (near dependency among various columns of  $X'X$  or  $\det(X'X) \approx 0$ ), the OLS method fails due to the problem of multicollinearity.

$$\phi = (Y - XB)'(Y - XB) \quad (2.4)$$

$$= Y'Y - B'X'Y - Y'XB - B'X'XB \quad (2.5)$$

$$\phi = Y'Y - 2B'X'Y + B'X'XB \quad (2.6)$$

Substituting  $\hat{Y} = X\hat{\beta}$  and  $E(B) = \hat{\beta}$ , into equation (2.6), we have the residual sum of squares

the results concerning the various parameters in terms of MSE and the conclusion is presented in section 5.

## 2.0 Methodology

Here we present the proposed estimators and briefly describe the background methods of Hoerl and Kennard (1970), Hoerl et al (1975), Khalaf and Shurkuri (2005) and Muniz et al (2010). However, the proposed ridge estimators will also be presented.

### 2.1 The Ridge Estimator

In a multiple linear regression model, one considers the model:

(2.1)

$$\begin{pmatrix} x_{11} & \dots & \dots & x_{1p} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x_{n1} & \dots & \dots & x_{np} \end{pmatrix}, \quad \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_p \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix} \text{ and } \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

The idea in ridge regression method is to add a small positive number ( $k > 0$ ) to the diagonal elements of the matrix  $(X'X)$  in order to obtain a ridge estimator that shortens the length of the regressor vector.

Let  $B$  be any estimate of the vector  $\beta$ . Let  $\phi$  be the residual sum of squares (RSS). Then

$$\phi = (Y - X\hat{\beta})'(Y - X\hat{\beta}) + (B - \hat{\beta})'X'X(B - \hat{\beta}) \quad (2.7)$$

$$\phi = \phi_{\min} + \phi(B) \quad (2.8)$$

where  $\phi_{\min} = (Y - X\hat{\beta})'(Y - X\hat{\beta})$  and  $\phi(B) = (B - \hat{\beta})'X'X(B - \hat{\beta})$

At  $B = 0$ ,  $\phi_0 = \phi_{\min} + \phi(0)$  where  $\phi(0) = \hat{\beta}'X'X\hat{\beta} > 0$  is the increment. However, the average distances from  $\hat{\beta}$  to  $\beta$  are large if the eigenvalue of the matrix  $X'X$  are small. The more the ill-conditioning of  $X'X$  the more the length of  $\hat{\beta}$  is further from  $\beta$  without appreciable increase in the residual sum of squares value. It seems reasonable if one moves from the minimum sum of squares value to the direction that will shorten the length of the regression vector  $\hat{\beta}$ . That is for a fixed value of  $\phi$  a single value of  $B$  is chosen, to minimize the length  $B'B$ . Thus,

Min  $B'B$

subject to

$$(B - \hat{\beta})'X'X(B - \hat{\beta}) = \phi_0, \quad \phi_0 > 0, \quad B \geq 0 \quad (2.9)$$

Converting equation (2.9) to unconstrained problem, we have

$$\text{Min } F = B'B + \frac{1}{k} \left[ (B - \hat{\beta})'X'X(B - \hat{\beta}) - \phi_0 \right] \quad (2.10)$$

where  $\frac{1}{k}$  is a multiplier. Then, a minimum value of  $B$  is attained if

$$\frac{\partial F}{\partial B} = 2B + \left( \frac{1}{k} \right) [2(X'X)B - 2(X'X)\hat{\beta}] = 0$$

Hence,

$$B = \hat{\beta}(k) = [X'X + kI]^{-1}X'Y \quad (2.11)$$

where  $k$  is chosen to satisfy the constraint in (2.9) by a process known as ridge regression estimation. The constant  $k$  ( $k > 0$ ) is known as a "RIDGE" or biased parameter, and is estimated from the observed data (Newhouse and Oman, 1971).

Hoerl and Kennard (1970), in their pioneer work showed that as  $k$  increases from zero to infinity, the regression estimate tends to zero (this process is known as shrinkage).

These estimates  $\hat{\beta}(k)$ , for certain values of  $k$ , yield biased results and minimum mean square error  $MSE(\hat{\beta}(k))$ . However, the  $MSE(\hat{\beta}(k))$  depends on the unknown parameters  $k$ ,  $\beta$  and  $\sigma^2$ .

The MSE of the biased estimator  $\hat{\beta}(k)$  is:

$$MSE(\hat{\beta}(k)) = Var(\hat{\beta}(k)) + [Bias(\hat{\beta}(k))]^2 \quad (2.12)$$

$$= \sigma^2 (\tilde{X}\tilde{X} + kI_p)^{-1} (\tilde{X}\tilde{X}') (\tilde{X}\tilde{X} + kI_p)^{-1} + k^2 \beta' (\tilde{X}\tilde{X} + kI_p)^{-2} \beta \quad (2.13)$$

where  $\tilde{X}$  is the centered and scaled values of  $X$ .

Since  $\tilde{X}\tilde{X}$  is a positive definite symmetric matrix, there exist an orthogonal matrix  $D$  such that  $D'CD = d$ , where  $d = diag(\lambda_1, \lambda_2, \dots, \lambda_p)$  and  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p$  are the eigenvalues of  $C = \tilde{X}\tilde{X}$ . Let  $\alpha = D'\beta$ , then equation (2.1) after centering and scaling becomes;

$$\tilde{Y} = \tilde{X}^* \alpha + \varepsilon \quad (2.14)$$

where  $\tilde{X}^* = \tilde{X}D$ , and  $\alpha = D'\beta$ ,  $\alpha$  is  $p \times 1$  vector and  $\beta$  is also a  $p \times 1$  vector of regression coefficients.

The total mean square error (TMSE) of the regression coefficient  $\hat{\beta}(k)$  in the presence of multicollinearity is given by:

$$TMSE(k) = E\left\{(\hat{\beta}(k) - \beta)'(\hat{\beta}(k) - \beta)\right\} = \sigma^2 \sum_{j=1}^p \frac{\lambda_j}{(\lambda_j + k)^2} + \sum_{j=1}^p \frac{k^2 \alpha_j^2}{(\lambda_j + k)^2} \quad (2.15)$$

where  $\alpha_j$  is the  $j$ th element of the vector  $\alpha = D'\beta$ . Instead of choosing only a single value of  $k$  separate ridge parameters for each of the regression coefficient is considered making the ridge regression parameter a vector denoted by  $k = (k_1, k_2, \dots, k_p)$ .

Hence,

$$TMSE(k_i) = \sigma^2 \sum_{j=1}^p \frac{\lambda_j}{(\lambda_j + k_i)^2} + \sum_{j=1}^p \frac{k_i^2 \alpha_j^2}{(\lambda_j + k_i)^2}, \text{ where } i = 1, 2, \dots, p \quad (2.16)$$

The value of  $k$  is selected to obtain the corresponding values of regression coefficients. In the presence of multicollinearity, the ridge regression parameter vary as  $k$  increases from zero until stability is achieved. However, the value of  $k$  is directly related to the amount of bias introduced. Hence the smallest value of  $k$  for which the stability occurs is selected.

Consequently,

$$\hat{k}_i = \frac{\hat{\sigma}^2}{\alpha_j^2} \quad (2.17)$$

where

$$\hat{\sigma}^2 = \frac{\epsilon_i' \epsilon_i}{(n-p)} \text{ represents the error variance of the multiple regression and } \alpha_j \text{ is the } j\text{th element of } \alpha.$$

## 2.2 The Proposed Estimators

Here, we present some existing estimators and the proposed estimators.

### The Method of Hoerl and Kennard, (1970)

Hoerl and Kennard (1970) showed that the value of  $k$  that will minimize Equation (2.15) the total mean square error (MSE) of the ridge regression coefficient is

$$k_j = \frac{\sigma^2}{\alpha_j^2} \quad j = 1, 2, \dots, p \quad (2.18)$$

where  $\sigma^2$  represents the error variance of the model in equation (2.1),  $\alpha_j$  is the  $j$ th element of  $\alpha$ . However, the optimal value of  $k_j$  depends on the unknown  $\sigma^2$  and  $\alpha_j^2$ , and they must be obtained from the observed data. They further suggested to replace  $\sigma^2$  and  $\alpha_j^2$  by their corresponding unbiased estimators. That is;

$$\hat{k}_j = \frac{\hat{\sigma}^2}{\hat{\alpha}_j^2} \quad (2.19)$$

where  $\hat{\sigma}^2 = \frac{\epsilon_i' \epsilon_i}{(n-p)}$  represents the residual mean square estimate which is unbiased for  $\sigma^2$  and  $\hat{\alpha}_j$  is the  $j$ th element of  $\alpha$  ( $p \times 1$  vector).

Hoerl and Kennard (1970) suggests  $k$  to be single value that is obtained from the Equation (2.19) as follows:

$$k_1 = \frac{\hat{\sigma}^2}{\hat{\alpha}_{\max}^2} \quad (2.20)$$

where  $\hat{\sigma}_{\max}$  is the maximum value of  $\hat{\alpha}_j$ .

### The Method of Fixed Point

Hoerl et al (1975) suggested estimating the value of  $k$  by using the harmonic mean of  $\alpha_j^2$  of Equation (2.19) as follows:

$$k_2 = \frac{p\hat{\sigma}^2}{\sum_{j=1}^p \hat{\alpha}_j^2} \quad (2.21)$$

### The Method of Lawless and Wang (1976)

Lawless and Wang (1976) adopts Bayesian approach and assumes that  $\hat{\beta}(k)$  has a prior distribution that is multivariate normal with mean 0 and variance-covariance matrix  $\sigma_\beta^2 I_p$ . They modified the methods of Hoerl and Kennard (1970) by including the eigenvalues of the design matrix in equation (2.21) to obtain the following estimators:

$$k_3 = \frac{p\sigma^2}{\sum_{j=1}^p \lambda_j \hat{\alpha}_j^2} \quad (2.22)$$

where  $\lambda_j$  is the eigenvalue of  $X^T X$

### The Method of Khalaf and Shukur (2005)

Khalaf and Shukur (2005) modified Lawless and Wang (1975) to obtain an improved method of estimating  $k$  that gives a smaller TMSE as follows

$$k_4 = \frac{\lambda_{\max} \hat{\sigma}}{(n-p)\hat{\sigma}^2 + \lambda_{\max} \hat{\alpha}_{\max}^2}$$

Muniz et al (2010) modified Hoerl and Kennard (1970) and Khalaf and Shukur (2005) to obtain the following estimators:

$$(a) k_5 = \max\left(\frac{1}{q_j}\right) \quad (2.23)$$

$$(b) k_6 = \max(q_j) \quad (2.24)$$

$$(c) k_7 = \left(\prod_{j=1}^p \frac{1}{q_j}\right)^{\frac{1}{p}} \quad (2.25)$$

$$(d) k_8 = \left(\prod_{j=1}^p q_j\right)^{\frac{1}{p}} \quad (2.26)$$

$$(e) k_9 = \text{median}\left(\frac{1}{q_j}\right) \quad (2.27)$$

$$\text{where } q_j = \frac{\lambda_{\max} \hat{\sigma}^2}{(n-p)\hat{\sigma}^2 + \lambda_{\max} \alpha_j^2}$$

### 2.3 Ridge Parameter Estimation When the Eigenvalues of the Matrix $\tilde{X}'\tilde{X}$ are Skewed

Skewness is the degree of asymmetry or departure from symmetry of a distribution or a set of data. (Spiegel and Stephen 2008). For a skewed set of data, the median tends to lie on the same side of the longer tail. Skewness (skew(y)) can be estimated as follows:

Given a set of data  $y_1, y_2, \dots, y_n$  skewness is:

$$skew(y) = \frac{\sum_{j=1}^p (y_j - \bar{y})^3}{(p-1)s^3} \quad (2.28)$$

where  $\bar{y}$  is the mean of the eigenvalues,  $s$  is the standard deviation of the eigenvalues and  $p$  is the number of explanatory variables. Considering the situation where the eigenvalues are skewed positively and the maximum  $\lambda_{max}$  is at the longer tail of the distribution it does not represent the distribution of the eigenvalues. In view of this, the centre of the distribution (the median) which gives a true representation of the eigenvalues (Szekely and Mori 2001, von Hippel 2005), is adopted in Khalaf and Shukur (2005).

Assuming that the model in equation (2.14) is observed to be in canonical form

$$Y = X^* \alpha + \varepsilon \quad (2.29)$$

Where  $X^* = XD$ , and  $\alpha = D'\beta$   $\tilde{X}'\tilde{X}$  is positive definite symmetric since  $D'D = DD' = I$  ( $p \times p$  orthogonal matrix),  $\alpha$  is  $p \times 1$  vector and  $\beta$  is also a  $p \times 1$  vector of regression coefficients and  $D'\tilde{X}'\tilde{X}D = diag(\lambda_1, \dots, \lambda_p)$  where  $(\lambda_1, \dots, \lambda_p)$  are the eigenvalues of  $\tilde{X}'\tilde{X}$ .

$D$  is the  $p \times p$  matrix of the eigenvectors corresponding to the eigenvalues of  $\tilde{X}'\tilde{X}$ .

In a situation where the explanatory variables are fixed, then the MSE is minimized when  $\beta$  is normalized eigenvectors corresponding to the largest eigenvalues of  $\tilde{X}'\tilde{X}$  subject to the constraint that  $\beta'\beta = 1$  (Newhouse and Oman 1971), and the columns of the eigenvectors are orthogonal (Ayres, 1962). Equally, the performance of the maximum and minimum eigenvalues does not differ in the estimation of the ridge parameter as attested by Kibria (2003) and Pasha and Ali Sha (2004). As well Khalaf and Shukur (2005) used the maximum eigenvalue which yields the maximum eigenvectors to estimate the ridge parameter  $k$  that yields a

$$P\{X > m\} = P\{X < m\} = \frac{1}{2} \quad (2.29)$$

Let  $X$  have exponential distribution with parameter  $\frac{1}{\lambda}$ . Then  $X$  has a density function

$$f_x(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{1}{\lambda}x} & x \geq 0 \\ 0 & elsewhere \end{cases} \quad (2.30)$$

The median is as in equation (2.29)

Since  $\lambda$  is the mean of this exponential distribution and for eigenvalues of the matrix  $\tilde{X}'\tilde{X}$  the mean is 1. Therefore the median when the mean is 1 is

$$m = \ln 2 \quad (2.31)$$

Substituting equation (2.31) in Khalaf and Shukur (2005) the following is obtained (Uzuke et al, 2015)

smaller MSE. Following Newhouse and Oman (1971), using the smallest eigenvalue will equally result to smaller MSE. Therefore in a case of skewed eigenvalues, their performance may not be the same. In view of this and in order to overcome this, the center is considered and the measurement of the center in a skewed set of data is the median. A skewed eigenvalue with skewness value of 2 is exponentially distributed as attested by Schmid and Makalic, (2009).

The median of a continuous random variable with cumulative distribution function  $F(x)$  is the value  $m$  (Ross 2009):

$$w_j = \frac{\ln 2(\hat{\sigma}^2)}{(n-p)\hat{\sigma}^2 + \ln 2(\hat{\sigma}_j^2)} \quad (2.32)$$

$j = 1, 2, \dots, p$

Using  $w_j$  in Muniz et al (2010), the following estimators for the ridge parameter  $k$  (Uzuke et al, 2015)

$$(a) k_{10} = \max\left(\frac{1}{w_j}\right) \quad (2.33)$$

$$(b) k_{11} = \max(w_j) \quad (2.34)$$

$$(c) k_{12} = \left( \prod_{j=1}^p \frac{1}{w_j} \right)^{\frac{1}{p}} \quad (3.35)$$

$$(d) k_{13} = \left( \prod_{j=1}^p w_j \right)^{\frac{1}{p}} \quad (2.36)$$

$$(e) k_{14} = \text{median}\left(\frac{1}{w_j}\right) \quad (2.37)$$

$$(f) k_{15} = \text{median}(w_j) \quad (2.38)$$

### 3 SIMULATION

The aim of this paper is to evaluate the MSE and the predictive ability of different RR and OLS estimators. This is done via Monte Carlo stimulation where 2000 replicates (R) are used. Here we give a brief description of the factors that vary in the simulations study.

#### The Number of Explanatory Variables (p)

In most studies, Lawless and Wang (1976), Khalaf and Shurkur (2005), Muniz et al (2010) and Uzuke et al (2015) used a smaller number of explanatory variables, here there is need to conduct an investigation with more number of explanatory variable to see which RR estimator is the best with respect to MSE. We are going to present for  $p = 3$  to 9.

#### The Sample Size

Here we have to be very cautious in choosing the number of observations because many combinations of different  $n$  and  $p$  will not be possible to estimate the degrees of freedom since the number of degrees freedom will exceed the no of explanatory variable. Here we choose small sample size  $n = 10$  and large sample size  $n = 80$ .

#### The Strength of Correlation among the Explanatory Variables

In order to achieve different degrees of collinearity among the explanatory variable, the explanatory variables were generated using:

$$x_{ij} = (1 - \rho^2)^{\frac{1}{2}} z_{ij} + \rho z_{ij} \quad (3.1)$$

Where  $\rho$  is the correlation between the explanatory variable and it is taken as  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$  and  $z_{ij}$  are generated using the standard normal distribution. More so the  $n$  observation for the independent variable are obtained using the following equation;

$$y_i = \beta_0 + \beta_1 x_{i1} + \dots + \beta_p x_{ip} + e_i \quad (3.2)$$

$i = 1, 2, \dots, n$  where  $e_i \sim N(0, \sigma^2)$  and  $\beta_0 = 0$ .

$\beta_j$  are chosen according to Newhouse and Oman (1971) that if the MSE is a function of  $\beta$ ,  $\sigma$  and  $k$  and if the explanatory variables are fixed, then the MSE is minimized when we choose this coefficient

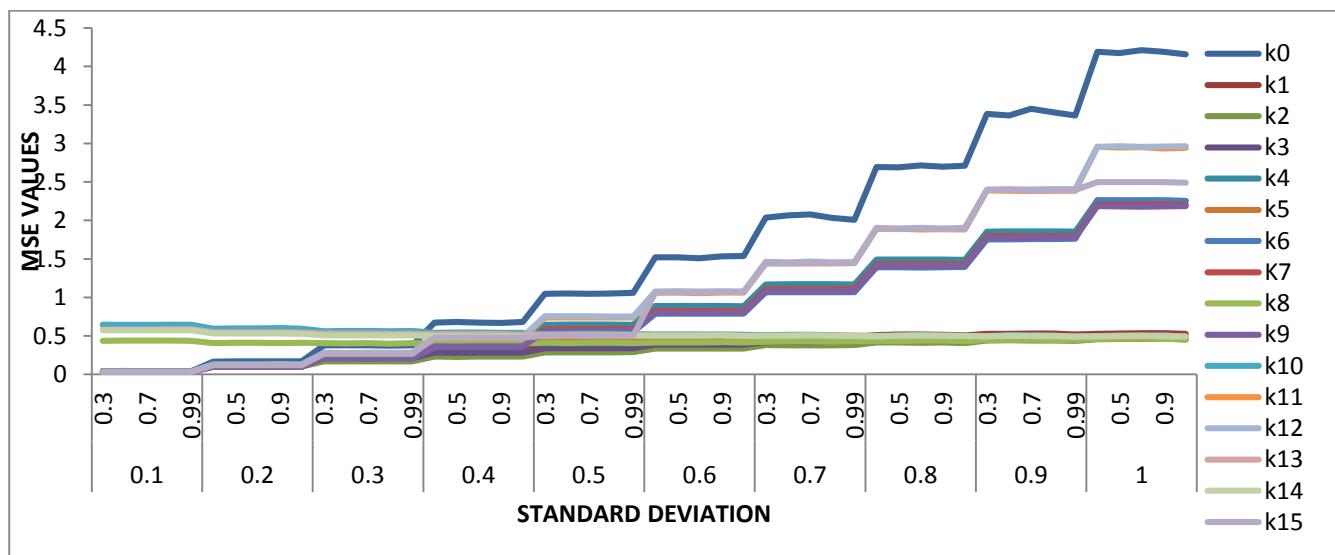
vectors  $\sum_{j=1}^p \beta_j^2 = 1$ . The variables  $X$  and  $Y$  are then

centered and scaled so that  $X'X$  and  $X'Y$  are in correlation form.

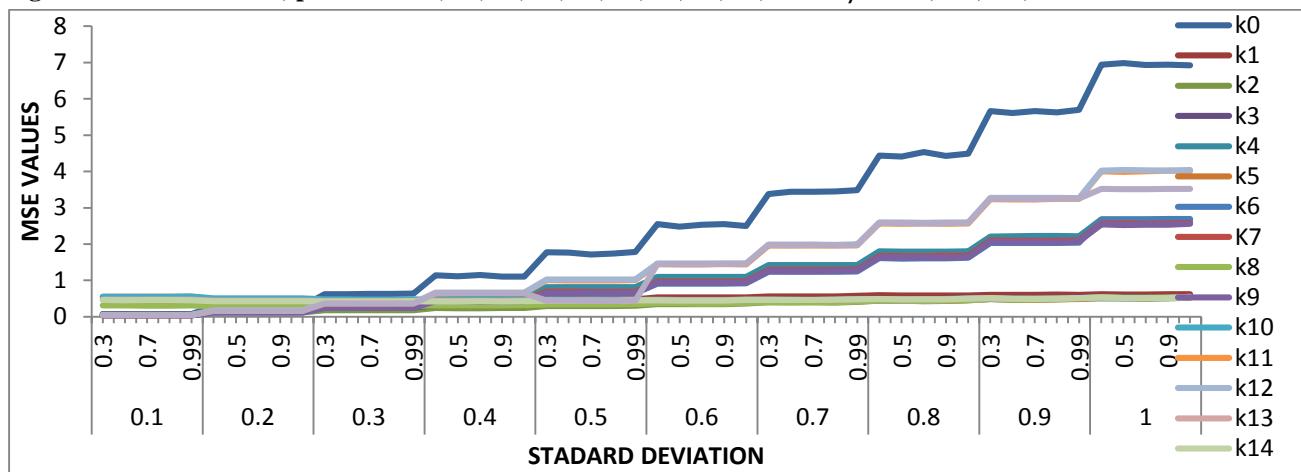
#### The Error Term $\sigma$

Kibria (2003) has shown that increasing the variance of  $e_i$  negatively affects the MSE especially that of OLS, on this note we are going to use  $\sigma$  as 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1.0.

The set of explanatory variables are generated for given values of  $n$ ,  $p$ ,  $\rho$ ,  $\beta$  and  $\sigma$ . Then the simulation was repeated 2,000 times and the mean MSE obtained. This simulation was carried out using a program written in R and the graph of the simulated MSE values were presented in Figures 1 to Figure 14.



**Figure 1** MSE for  $n=10$ ,  $p=3$   $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$



**Figure 2** MSE for  $n=10$ ,  $p=4$   $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

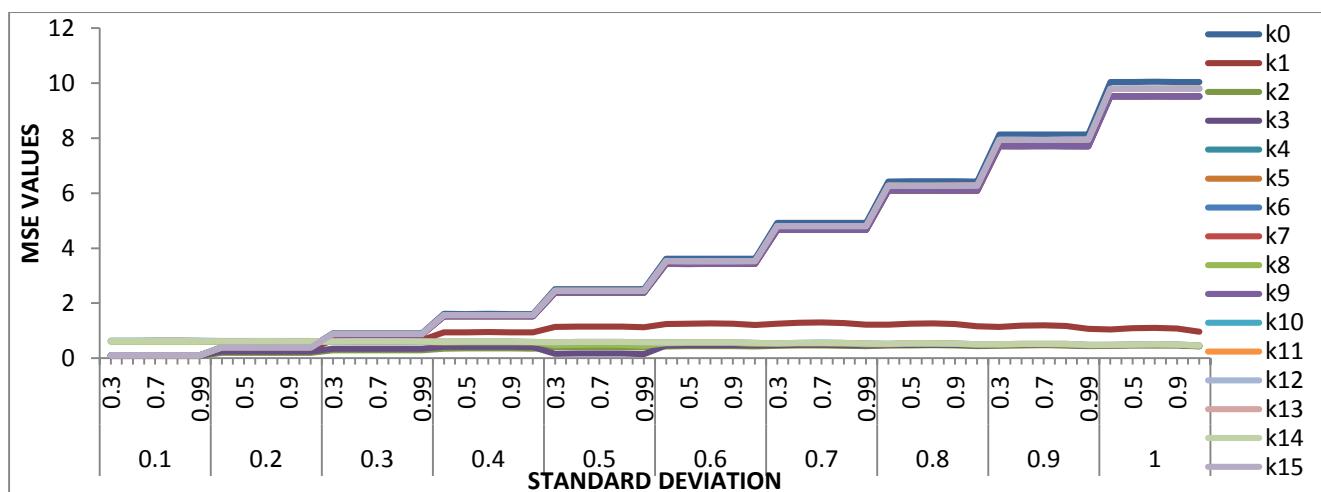


Figure 3 MSE for  $n=10, p=5 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

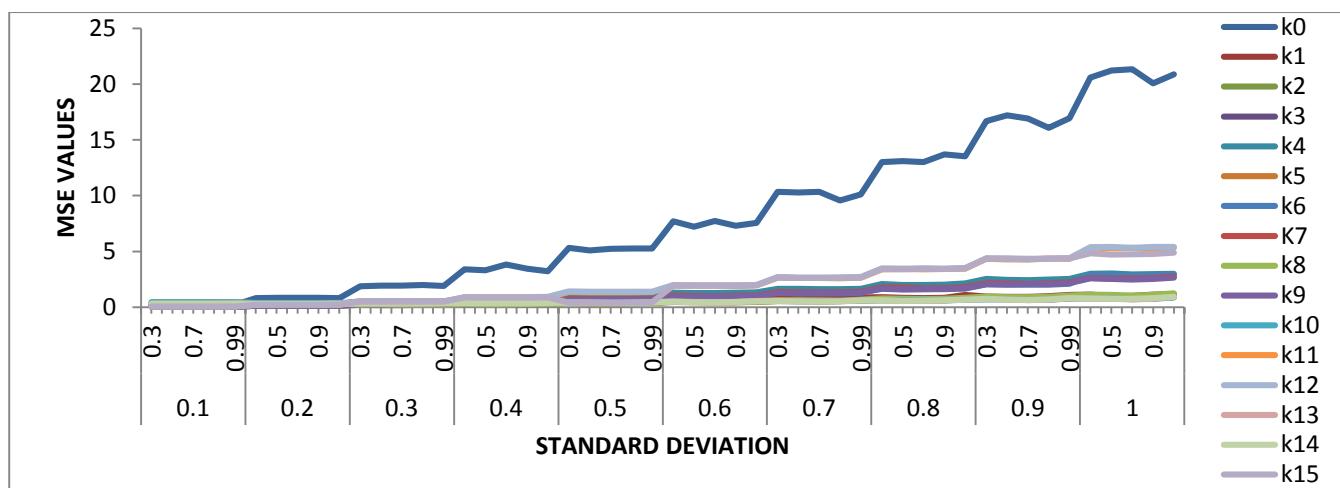


Figure 4 MSE for  $n=10, p=6 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

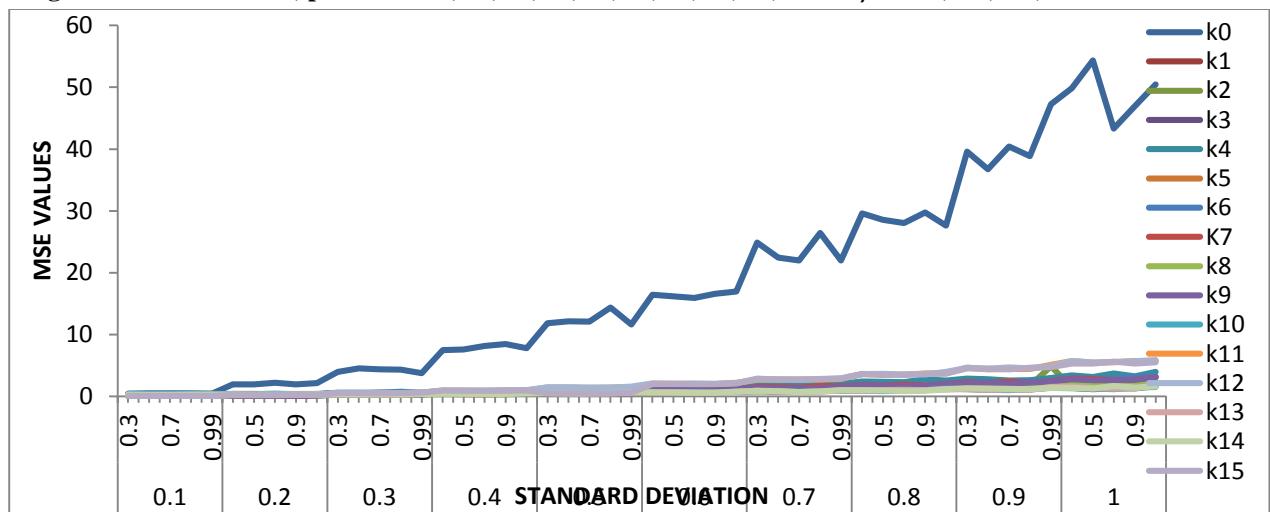


Figure 5 MSE for  $n=10, p=7 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

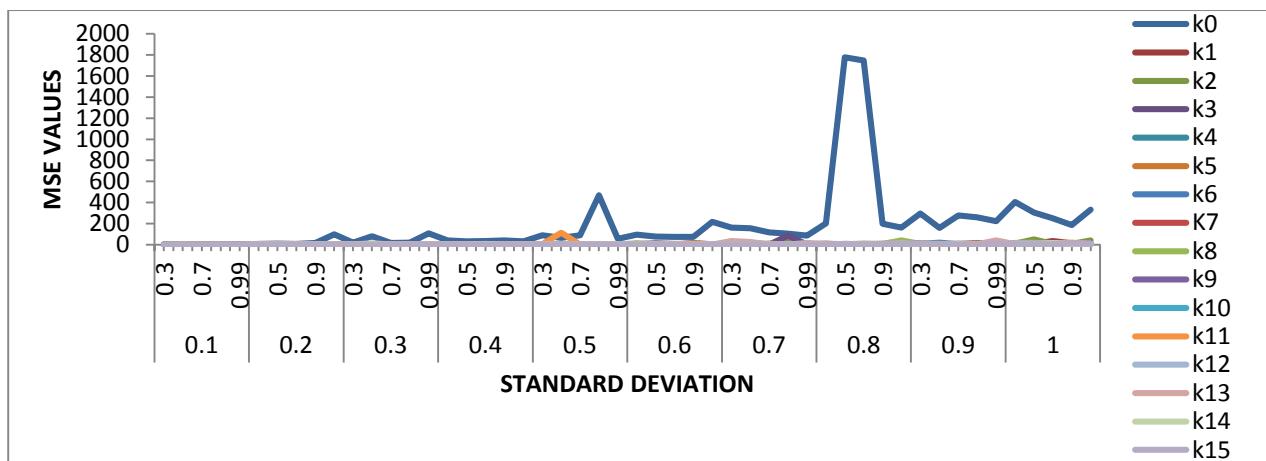


Figure 6 MSE for  $n=10, p=8 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

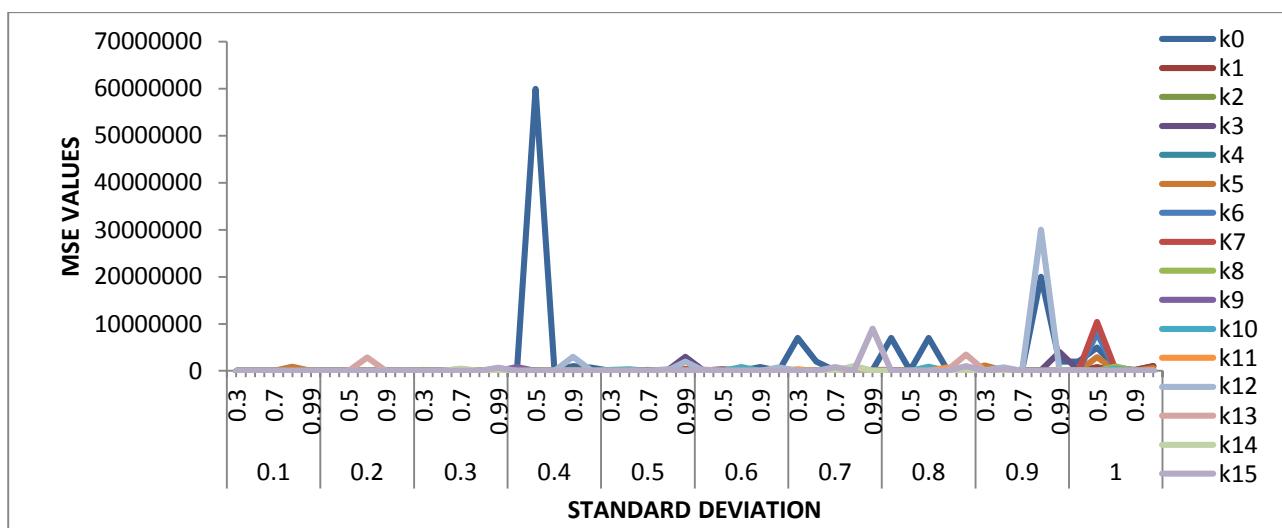


Figure 7 MSE for  $n=10, p=9 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

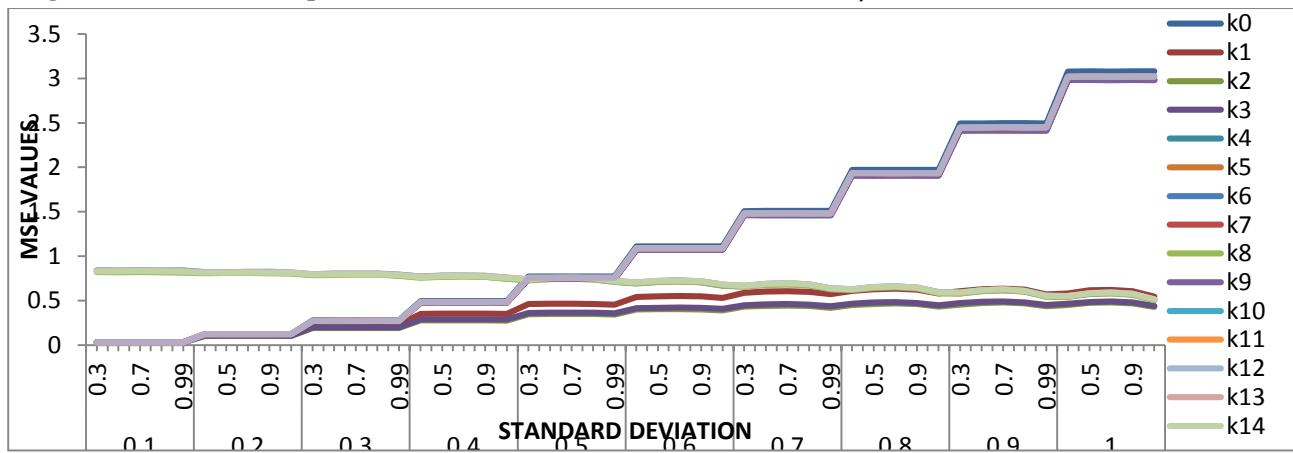


Figure 8 MSE for  $n=80, p=3 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

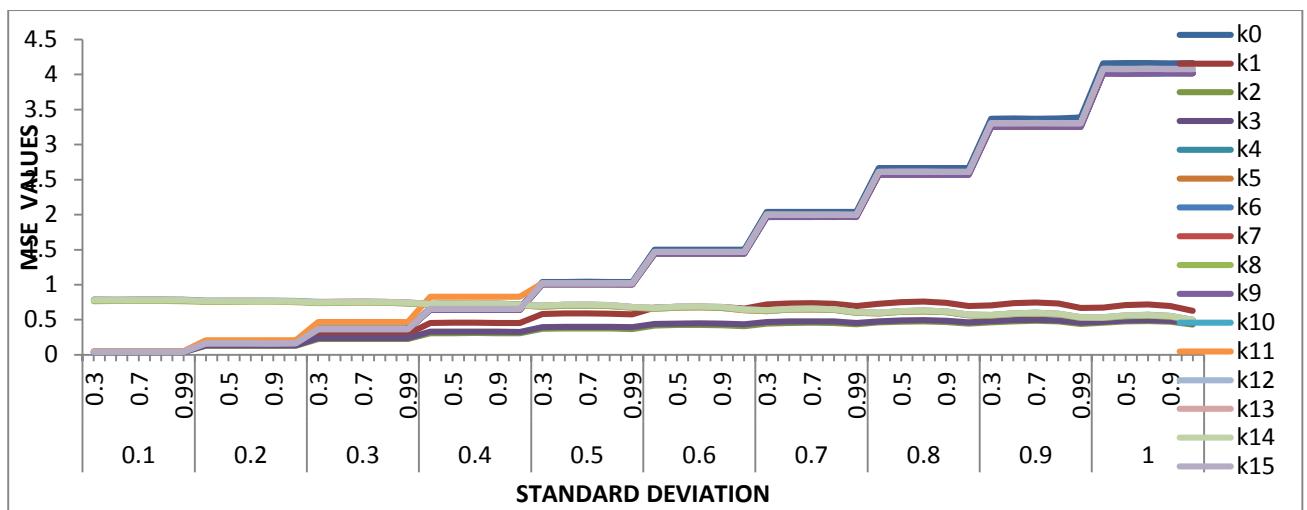


Figure 9 MSE for  $n=80, p=4 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

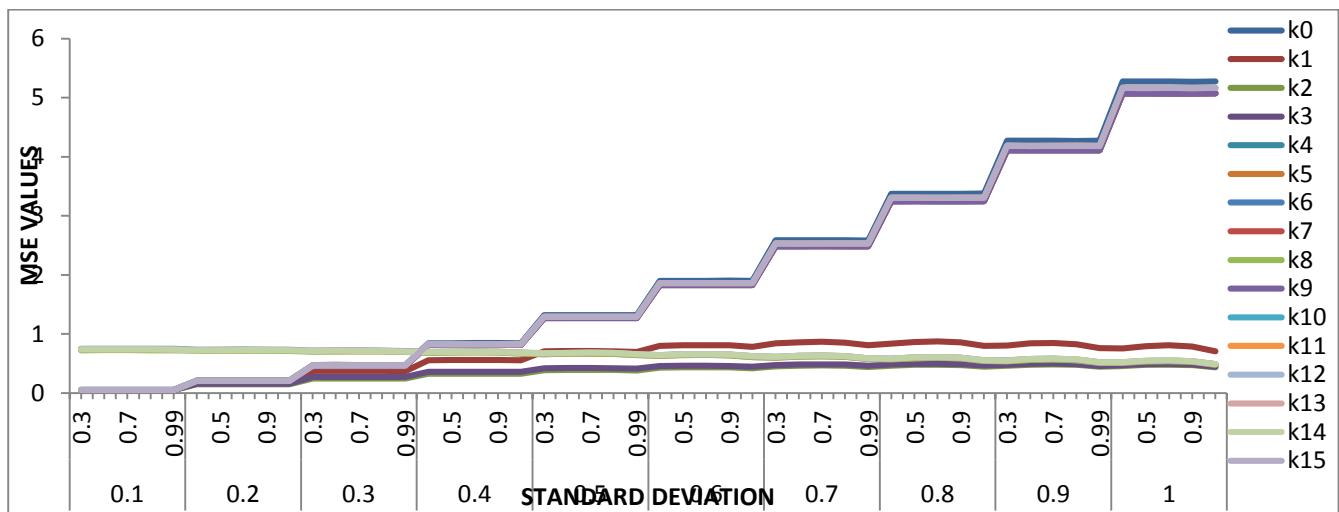


Figure 10 MSE for  $n=80, p=5 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

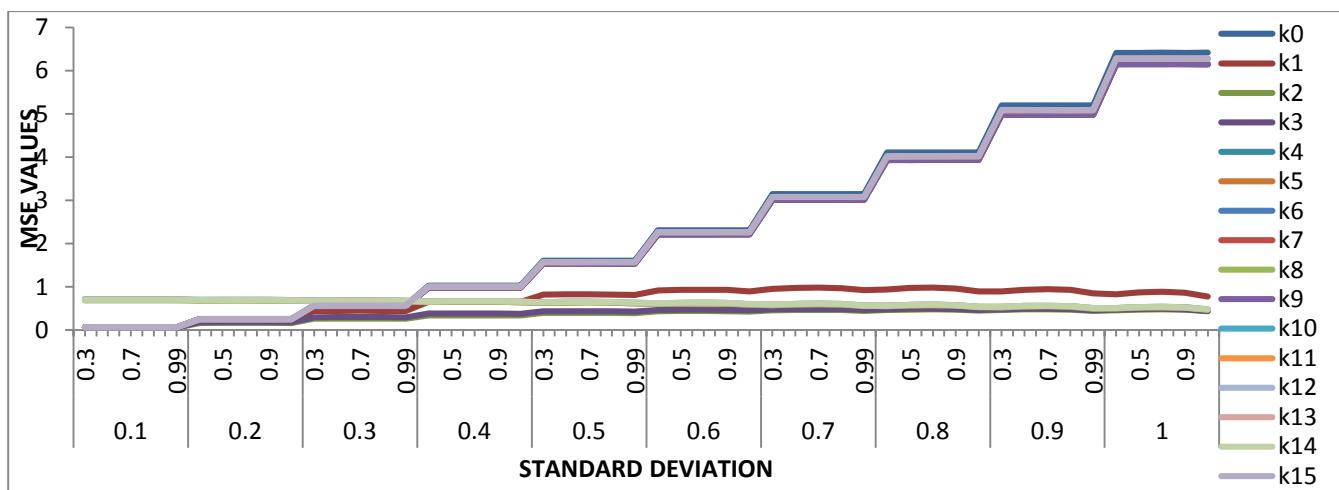


Figure 11 MSE for  $n=80, p=6 \sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

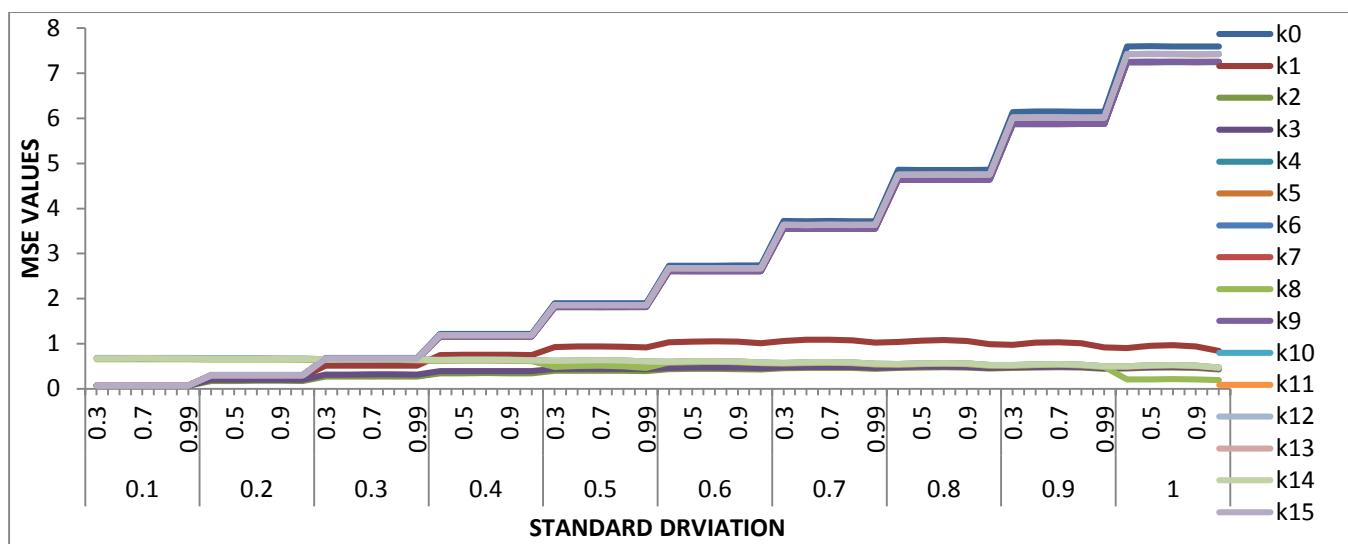


Figure 12 MSE for  $n=80$ ,  $p=7$   $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

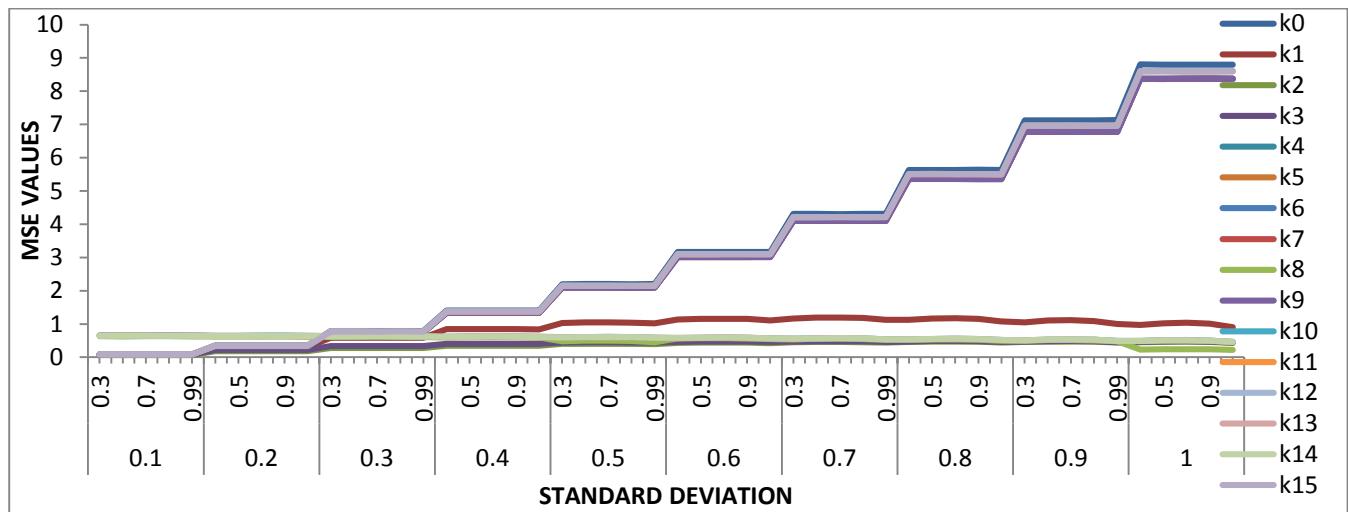


Figure 13 MSE for  $n=80$ ,  $p=8$   $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

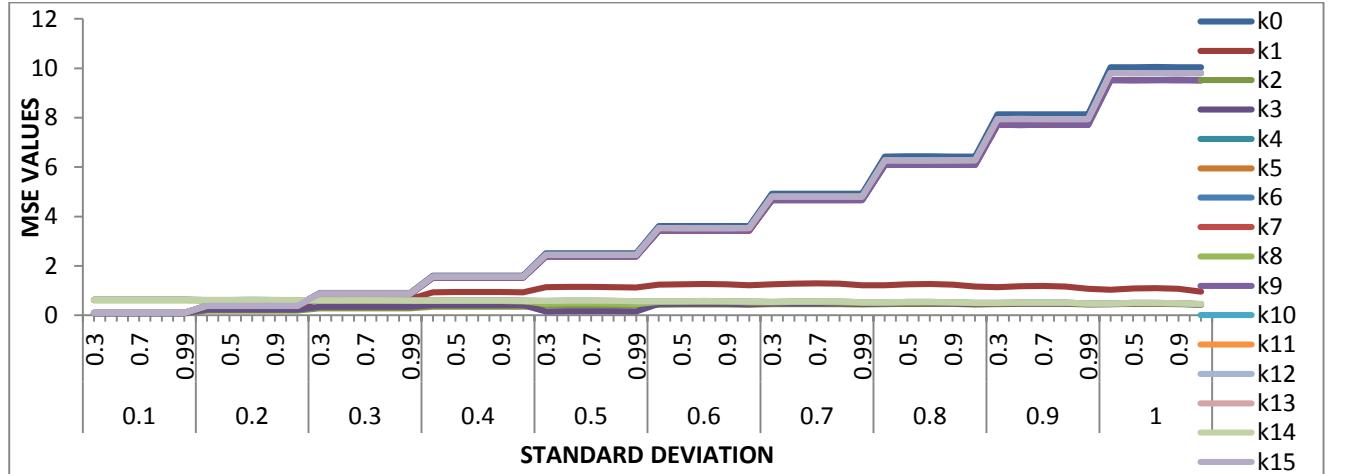


Figure 14 MSE for  $n=80$ ,  $p=9$   $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

#### 4. Discussions

Here we present the result of the Monte Carlo simulation concerning the MSEs of the different existing and proposed estimators compared to OLS. A conventional approach to report the result of a Monte Carlo simulation is to present the values of the MSE under different conditions in a table. When determining the way of presentation, some account has to be taken to the result obtained. Originally, it was intended to present the results obtained in form of tables but the results are too extensive, presenting these results in forms of a table makes it difficult to follow the head lines of the findings. Hence, the findings are presented in form of figures which summarises the result with respect to different factors under investigation. More exact results of the simulated MSEs for the 15 estimators under investigations are presented in the Appendix. MSE values for  $n = 10$ ,  $p=3, 4, 5, 6, 7, 8$  and  $9$ ,  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$  were presented in Tables A1 to A7 in the Appendix A and MSE values for  $n = 80$ ,  $p=3, 4, 5, 6, 7, 8$  and  $9$ ,  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$  were presented in Tables B1 to B7 in Appendix B.

The result show that as standard deviation increases, the MSE of various estimators increases, also as the number of explanatory variable  $p$  increases the MSE also increases, and for all values of the standard deviation  $\sigma$  the ridge estimators have smaller values of MSE than the OLS method. However, the performance of the proposed estimators  $k_{10}$ ,  $k_{12}$  and  $k_{15}$  in terms of MSE is better than the rest of the considered estimators.

As a function of the correlation coefficient  $\rho$  between the explanatory variables for sample sizes  $n = 10$  and  $n = 80$  as shown in Figures 1 – 14, and for smaller  $\sigma$ , the correlation coefficient has no significant effect on the MSE. But when  $\sigma$  increases with increase in the correlation coefficient, there is increase in the values of MSE of the estimators. MSE is on the high side when  $\sigma$  is high and the sample size  $n$  is small (see Tables A and B in the Appendix).

When the explanatory variables are multicollinear, all the considered ridge estimators have smaller MSE than the method of OLS. Moreover, when the sample size increases, the MSE decreases for smaller number of explanatory variables  $p$ , but for large  $p$  the MSE increases. Consequently, as  $n$  and  $p$  increases, the estimators  $k_2$ ,  $k_3$ ,  $k_5$ ,  $k_6$ ,  $k_7$ ,  $k_9$ ,  $k_{11}$ ,  $k_{12}$  and  $k_{15}$  is better than the rest of the estimators when the  $\sigma$  is small.

#### 5.0 Conclusion

We have reviewed and proposed some new estimators in this paper under basic assumptions and their performances compared using MSE via simulation studies. The proposed estimators were defined based on the works of Lawless and Wang (1976) and Khalaf and Shukur (2005). It was observed that the performance of these estimators depend on the variance of the random error  $\sigma$ , the correlation among the explanatory variables  $\rho$ , the sample size  $n$  and the number of explanatory variables  $p$ . The increase in the number of explanatory variables and increase in the sample size reduces the MSE of the estimators even when the correlation between the explanatory variables are high, but for small sample size, MSE increases as the values of  $p$  increases. Consequently increase in  $\sigma$  increases the MSE of the estimators. That is better estimators increases in MSE as  $\sigma$  increases.

However, all the proposed and existing estimators performed better than OLS in terms of small MSE in all conditions considered in the simulation study. Particularly, the proposed estimators  $k_{11}$ ,  $k_{12}$  and  $k_{15}$  outperformed others and can be recommended.

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## Appendix A

**Table A.1 MSE n=10, p=9  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99**

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	0.0417	0.039	0.0354	0.0378	0.0393	0.6436	0.0331	0.0326	0.4356	0.032	0.6452	0.0327	0.0353	0.5826	0.5737	0.033
	0.5	0.0428	0.0392	0.0353	0.038	0.0389	0.6413	0.033	0.0328	0.4418	0.032	0.6447	0.0328	0.0354	0.5822	0.5733	0.0331
	0.7	0.0424	0.0392	0.0352	0.0383	0.0391	0.6407	0.0332	0.0328	0.4416	0.032	0.6449	0.0327	0.035	0.5825	0.5724	0.0329
	0.9	0.0424	0.0389	0.0354	0.0381	0.0392	0.6445	0.0331	0.0328	0.4393	0.0319	0.6448	0.0327	0.0355	0.5842	0.5721	0.033
	0.99	0.0416	0.039	0.0352	0.0381	0.0391	0.6428	0.0331	0.033	0.4354	0.032	0.6447	0.0327	0.0352	0.587	0.5746	0.033
0.2	0.3	0.1647	0.1314	0.1025	0.1219	0.1363	0.5818	0.0983	0.1121	0.4087	0.1005	0.5958	0.121	0.13	0.5489	0.5306	0.1219
	0.5	0.168	0.1312	0.1025	0.1214	0.1364	0.5822	0.0986	0.1119	0.4092	0.1005	0.5997	0.1213	0.1294	0.5476	0.5303	0.1221
	0.7	0.1688	0.1313	0.1023	0.1216	0.1366	0.5794	0.0984	0.112	0.4104	0.1005	0.5987	0.1215	0.1295	0.5482	0.5318	0.1225
	0.9	0.1681	0.1313	0.1025	0.1219	0.1376	0.5785	0.0989	0.1119	0.4085	0.1007	0.6028	0.1211	0.1303	0.5489	0.5298	0.1227
	0.99	0.1692	0.1311	0.1027	0.1222	0.1375	0.581	0.0991	0.1123	0.4105	0.1007	0.5938	0.1214	0.1285	0.5422	0.5271	0.1227
0.3	0.3	0.378	0.2364	0.1696	0.2095	0.2725	0.5216	0.2071	0.2303	0.4063	0.2113	0.5603	0.2678	0.2818	0.5327	0.5046	0.2697
	0.5	0.3784	0.2366	0.1684	0.2092	0.2724	0.5243	0.207	0.2304	0.4037	0.2103	0.5653	0.2681	0.2786	0.5315	0.5056	0.2691
	0.7	0.379	0.2372	0.1694	0.2088	0.2727	0.5226	0.2069	0.23	0.405	0.2101	0.5646	0.2681	0.2809	0.5334	0.5077	0.2695
	0.9	0.3738	0.2362	0.1689	0.2094	0.2736	0.5216	0.207	0.2302	0.3977	0.2104	0.5616	0.268	0.2815	0.5273	0.5076	0.2704
	0.99	0.379	0.2364	0.1695	0.2093	0.2726	0.5228	0.2075	0.2313	0.406	0.2117	0.5643	0.2679	0.2808	0.5289	0.5066	0.2699
0.4	0.3	0.6712	0.3297	0.2315	0.2831	0.4406	0.4834	0.3587	0.3877	0.4053	0.3642	0.5397	0.4741	0.4891	0.5214	0.4998	0.4772
	0.5	0.6819	0.3312	0.2299	0.2834	0.4405	0.4834	0.3578	0.3881	0.4005	0.3628	0.5423	0.4746	0.487	0.5212	0.4981	0.4782
	0.7	0.6715	0.331	0.2302	0.2836	0.4409	0.4865	0.3574	0.3875	0.4069	0.363	0.5432	0.4747	0.4857	0.5214	0.4953	0.4775
	0.9	0.6697	0.3305	0.2307	0.283	0.4417	0.4866	0.3578	0.3878	0.4059	0.364	0.539	0.4757	0.4878	0.521	0.4975	0.4769
	0.99	0.68	0.3293	0.2301	0.2829	0.4414	0.4777	0.3589	0.3882	0.4043	0.3659	0.538	0.4746	0.488	0.5159	0.4901	0.4783
0.5	0.3	1.0487	0.4033	0.287	0.34	0.6452	0.4562	0.552	0.5865	0.4098	0.5601	0.5245	0.7384	0.7556	0.515	0.5026	0.7424
	0.5	1.0523	0.4051	0.2855	0.3414	0.6469	0.4551	0.5516	0.5861	0.4088	0.5595	0.5255	0.7381	0.7532	0.5139	0.5079	0.7435
	0.7	1.0485	0.4064	0.284	0.3415	0.648	0.4595	0.5507	0.5865	0.4124	0.5584	0.5326	0.739	0.7569	0.5161	0.5081	0.742
	0.9	1.0499	0.405	0.286	0.3403	0.646	0.4591	0.5516	0.5861	0.4084	0.5571	0.5279	0.7397	0.7569	0.513	0.5058	0.7427
	0.99	1.0601	0.4011	0.2885	0.342	0.644	0.4535	0.5533	0.587	0.4071	0.5627	0.5229	0.7404	0.7532	0.5115	0.503	0.7434
0.6	0.3	1.5192	0.4562	0.3359	0.3876	0.8882	0.4487	0.789	0.8261	0.413	0.7997	0.5169	1.0657	1.0743	0.5096	0.5033	1.0656
	0.5	1.5209	0.4597	0.3358	0.3869	0.8882	0.451	0.789	0.8251	0.4164	0.7971	0.5162	1.0623	1.0795	0.5097	0.5036	1.0681
	0.7	1.5084	0.4601	0.3379	0.3866	0.8899	0.4482	0.7876	0.8238	0.4137	0.7966	0.5201	1.0566	1.0776	0.5117	0.5023	1.0644
	0.9	1.5325	0.4547	0.3343	0.3852	0.8886	0.4502	0.7884	0.8257	0.4175	0.7971	0.5206	1.0625	1.079	0.5104	0.5073	1.071
	0.99	1.5381	0.4507	0.3377	0.3867	0.8859	0.4451	0.7889	0.8264	0.4134	0.8018	0.5166	1.0646	1.0756	0.5012	0.4973	1.0699
0.7	0.3	2.036	0.4924	0.3802	0.4201	1.1696	0.445	1.0699	1.1063	0.4227	1.0816	0.51	1.4404	1.4636	0.5034	0.5006	1.4473
	0.5	2.064	0.4965	0.379	0.4236	1.1707	0.4415	1.0676	1.107	0.4256	1.0804	0.5157	1.4441	1.4545	0.5024	0.5018	1.4408
	0.7	2.0801	0.4994	0.3775	0.4203	1.1715	0.4462	1.0682	1.1067	0.4284	1.0796	0.5129	1.443	1.4659	0.5058	0.5072	1.447
	0.9	2.0337	0.4944	0.3791	0.4238	1.1713	0.4428	1.0673	1.1061	0.4242	1.0788	0.5096	1.4413	1.4546	0.506	0.502	1.4498
	0.99	2.0058	0.4836	0.3815	0.4216	1.167	0.4432	1.069	1.1088	0.4259	1.0851	0.5037	1.446	1.4584	0.5006	0.4977	1.4493
0.8	0.3	2.6934	0.5123	0.4164	0.4519	1.4902	0.4399	1.3936	1.4292	0.4354	1.4068	0.5025	1.8907	1.8986	0.4987	0.4951	1.902
	0.5	2.6884	0.5198	0.4133	0.4511	1.4912	0.4478	1.3913	1.4282	0.4334	1.4038	0.5041	1.8892	1.8959	0.5005	0.5039	1.8875
	0.7	2.7156	0.517	0.4128	0.4492	1.4917	0.4484	1.39	1.4299	0.4404	1.4052	0.5087	1.883	1.9054	0.5026	0.5049	1.8939
	0.9	2.6965	0.5149	0.4163	0.4529	1.4903	0.4469	1.3909	1.4309	0.4406	1.4061	0.5052	1.8853	1.8964	0.5034	0.5015	1.8875
	0.99	2.712	0.5074	0.4083	0.4491	1.4873	0.4406	1.3952	1.4289	0.4303	1.4111	0.5014	1.8818	1.9022	0.4962	0.4968	1.8911
0.9	0.3	3.3849	0.5257	0.44	0.4708	1.8528	0.4563	1.7555	1.7949	0.4485	1.7741	0.4969	2.3902	2.4006	0.4998	0.4952	2.3972
	0.5	3.3647	0.5274	0.4426	0.4758	1.8574	0.4532	1.7541	1.7945	0.4474	1.772	0.5105	2.3844	2.4013	0.4974	0.5019	2.4036
	0.7	3.4527	0.5317	0.4386	0.4745	1.857	0.4549	1.7571	1.7921	0.4457	1.7696	0.5044	2.3818	2.4035	0.5003	0.5001	2.3842
	0.9	3.4031	0.5305	0.4381	0.4725	1.8559	0.4536	1.7582	1.7961	0.4491	1.7753	0.5019	2.3858	2.406	0.5006	0.497	2.3935
	0.99	3.3635	0.5177	0.4357	0.4718	1.8544	0.4564	1.7603	1.7958	0.4462	1.7809	0.4926	2.385	2.4087	0.4905	0.4911	2.3931
1	0.3	4.1904	0.5293	0.457	0.4875	2.2628	0.4621	2.2598	2.2051	0.4595	2.1851	0.4974	2.9546	2.955	0.494	0.496	2.9426
	0.5	4.1747	0.5324	0.4622	0.4877	2.2602	0.4646	2.2581	2.2039	0.468	2.1834	0.5034	2.9484	2.9636	0.5022	0.4974	2.9583
	0.7	4.2111	0.5376	0.4606	0.4869	2.2605	0.4626	2.2579	2.203	0.4686	2.1792	0.503	2.9504	2.9565	0.5038	0.5028	2.9457
	0.9	4.1916	0.5366	0.4644	0.486	2.2611	0.4691	2.2556	2.2059	0.4605	2.1834	0.5054	2.9362	2.9579	0.501	0.4941	2.

Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

0.5	0.2761	0.1868	0.1173	0.1614	0.1976	0.4848	0.1156	0.1348	0.2914	0.1181	0.5083	0.1639	0.1771	0.4423	0.4291	0.1653	
0.7	0.2746	0.1866	0.1169	0.1616	0.1981	0.4832	0.1145	0.1348	0.2885	0.1179	0.504	0.1634	0.1779	0.4407	0.4274	0.1658	
0.9	0.2804	0.1865	0.1175	0.1615	0.1973	0.481	0.115	0.1349	0.2874	0.118	0.5069	0.1642	0.1779	0.4411	0.4257	0.1654	
0.3	0.2851	0.1857	0.1181	0.1615	0.197	0.4811	0.1159	0.1357	0.2967	0.1192	0.5067	0.1646	0.1785	0.4449	0.4285	0.1659	
0.5	0.2851	0.1857	0.1181	0.1615	0.197	0.4811	0.1159	0.1357	0.2967	0.1192	0.5067	0.1646	0.1785	0.4449	0.4285	0.1659	
0.3	0.62	0.3102	0.1808	0.2546	0.369	0.4291	0.2403	0.2735	0.3012	0.2462	0.4793	0.3639	0.3823	0.4359	0.4186	0.3656	
0.5	0.6183	0.3104	0.1787	0.2515	0.369	0.4297	0.2393	0.273	0.2971	0.246	0.4754	0.364	0.3806	0.4378	0.4191	0.3665	
0.7	0.6255	0.3107	0.1778	0.252	0.3688	0.425	0.2401	0.2722	0.2938	0.2443	0.4735	0.3643	0.3835	0.4334	0.4156	0.3658	
0.9	0.6276	0.3102	0.1789	0.2518	0.3674	0.4286	0.2403	0.2733	0.2985	0.2452	0.4741	0.3636	0.3825	0.4325	0.4181	0.3659	
0.9	0.6355	0.3104	0.1826	0.2546	0.3699	0.4284	0.2416	0.2749	0.3023	0.2483	0.4775	0.3641	0.3833	0.4365	0.4226	0.3681	
0.4	0.3	1.1383	0.4087	0.2386	0.3213	0.5718	0.3956	0.4159	0.4566	0.3155	0.4246	0.6425	0.6659	0.4389	0.4218	0.649	
0.5	1.113	0.409	0.237	0.3192	0.5715	0.3943	0.4141	0.4562	0.3154	0.4232	0.4589	0.6431	0.6657	0.4388	0.4201	0.6473	
0.7	1.1508	0.4094	0.2357	0.3177	0.5747	0.3941	0.4146	0.4573	0.3148	0.422	0.4606	0.6446	0.6668	0.4352	0.424	0.6472	
0.9	1.1065	0.4099	0.2398	0.3188	0.5729	0.3933	0.4153	0.4573	0.3152	0.4218	0.4646	0.6443	0.6631	0.4355	0.4199	0.6498	
0.9	1.1051	0.411	0.2409	0.3218	0.5715	0.4001	0.4199	0.4626	0.3238	0.429	0.4641	0.646	0.6656	0.4399	0.427	0.6494	
0.5	0.3	1.7766	0.4793	0.2966	0.3724	0.8146	0.3862	0.6423	0.6916	0.3404	0.6545	0.462	1.0066	1.0261	0.444	0.4415	1.0074
0.5	1.7664	0.4816	0.2923	0.3701	0.8128	0.3883	0.637	0.6898	0.3351	0.6506	0.4577	1.0007	1.0255	0.4429	0.4395	1.0116	
0.7	1.7168	0.4823	0.2947	0.3698	0.8137	0.3818	0.6379	0.6841	0.3404	0.6522	0.4595	0.0011	1.0275	0.4433	0.4416	1.0101	
0.9	1.7382	0.481	0.2947	0.3721	0.8122	0.3843	0.6388	0.6858	0.3387	0.6493	0.4602	1.0038	1.0258	0.4459	0.4386	1.0109	
0.9	1.7847	0.4828	0.3019	0.3777	0.8123	0.3865	0.6437	0.6923	0.3439	0.6614	0.4618	1.0058	1.0298	0.4497	0.4496	1.0121	
0.6	0.3	2.5512	0.5299	0.3511	0.4146	1.097	0.3942	0.9155	0.9661	0.3751	0.9312	0.4676	1.4448	1.4701	0.4537	0.4544	1.4527
0.5	2.4815	0.5297	0.3445	0.4109	1.0951	0.3907	0.9106	0.9661	0.3698	0.9304	0.4635	1.4442	1.4653	0.4537	0.4477	1.4468	
0.7	2.5362	0.5286	0.345	0.4146	1.0967	0.3916	0.9101	0.9667	0.3601	0.9273	0.4613	1.4417	1.4657	0.4488	0.4455	1.4469	
0.9	2.5508	0.5295	0.3497	0.4114	1.0953	0.3922	0.9132	0.9659	0.3678	0.9288	0.4639	1.4447	1.4659	0.4541	0.4466	1.4565	
0.9	2.4948	0.5301	0.3607	0.4223	1.0928	0.3996	0.9207	0.9736	0.3785	0.9374	0.4713	1.4445	1.4697	0.4703	0.4629	1.4585	
0.7	3.3831	0.5597	0.4045	0.4557	1.4232	0.4173	1.2402	1.2933	0.3982	1.2615	0.4734	1.9641	1.984	0.4716	0.4661	1.9799	
0.5	3.4419	0.5574	0.3976	0.4512	1.4201	0.4077	1.2379	1.2876	0.3924	1.2555	0.4737	1.9596	1.9861	0.4645	0.4654	1.9656	
0.7	3.4376	0.5611	0.3974	0.448	1.4197	0.4055	1.2386	1.2872	0.4016	1.2522	0.4635	1.9587	1.9847	0.4651	0.4579	1.9745	
0.9	3.4541	0.562	0.3948	0.446	1.4255	0.4099	1.2365	1.2926	0.3967	1.2626	0.4712	1.963	1.981	0.4659	0.4666	1.9729	
0.9	3.4811	0.5732	0.4123	0.4618	1.4198	0.4266	1.2456	1.2983	0.4187	1.2728	0.4888	1.9717	1.9948	0.4792	0.4764	1.9694	
0.8	0.3	4.4402	0.5929	0.4479	0.4915	1.7982	0.4539	1.6144	1.6708	0.4429	1.6366	0.4885	2.5569	2.5965	0.4865	0.4864	2.5823
0.5	4.4135	0.5829	0.4362	0.4801	1.7934	0.4418	1.6022	1.6724	0.4334	1.6324	0.4835	2.5563	2.594	0.4743	0.4775	2.5748	
0.7	4.5383	0.5852	0.4323	0.4753	1.7965	0.4299	1.608	1.6663	0.4353	1.6299	0.4843	2.5568	2.5886	0.4764	0.4757	2.5722	
0.9	4.4288	0.5814	0.4372	0.479	1.7926	0.4375	1.6057	1.6754	0.4421	1.6283	0.4884	2.5623	2.5913	0.4815	0.4822	2.5761	
0.9	4.4859	0.5878	0.4507	0.4992	1.8008	0.4588	1.6227	1.6763	0.4561	1.6429	0.5042	2.5683	2.5961	0.4967	0.4963	2.5897	
0.9	5.6607	0.6062	0.4864	0.5145	2.2112	0.4903	2.0334	2.1007	0.4919	2.0622	0.5022	3.2387	3.273	0.5016	0.5123	3.2494	
0.5	5.6071	0.599	0.4638	0.5086	2.2188	0.4874	2.035	2.0915	0.4746	2.0548	0.4988	3.2328	3.2721	0.4915	0.4988	3.247	
0.7	5.663	0.6004	0.4717	0.4988	2.2242	0.4691	2.0294	2.0913	0.4765	2.0554	0.4951	3.2329	3.2721	0.4952	0.4987	3.2389	
0.9	5.6307	0.6084	0.4783	0.508	2.2205	0.4825	2.0325	2.0957	0.4876	2.0525	0.5032	3.2457	3.2764	0.4938	0.4983	3.2529	
0.99	5.7008	0.6043	0.4946	0.528	2.2155	0.4918	2.0425	2.1037	0.5023	2.0741	0.5109	3.2492	3.2624	0.5108	0.5106	3.2563	
1	0.3	6.9375	0.6232	0.5104	0.546	2.6835	0.5264	2.6742	2.5686	0.5382	2.5446	0.5185	4.0073	4.0269	0.5239	0.5308	4.0352
0.5	6.9825	0.6123	0.5081	0.5441	2.6864	0.5176	2.6771	2.5646	0.5249	2.5277	0.5094	3.9901	4.0397	0.5132	0.5167	4.0263	
0.7	6.936	0.6138	0.4964	0.5346	2.6844	0.5054	2.6758	2.564	0.5235	2.5252	0.519	4.0041	4.029	0.5139	0.5153	4.0036	
0.9	6.9408	0.6222	0.5082	0.5286	2.6902	0.5215	2.6742	2.5661	0.5324	2.5298	0.5147	4.0212	4.0208	0.5159	0.5179	4.0188	
0.99	6.9248	0.6198	0.5218	0.5498	2.6897	0.552	2.6615	2.5833	0.5445	2.5564	0.5378	4.0157	4.0397	0.553	0.5432	4.0238	

Table A.3 MSE n=10, p=5  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	
0.1	0.3	0.1167	0.0892	0.059	0.0817	0.0917	0.4817	0.0447	0.0477	0.2058	0.0443	0.4848	0.0527	0.0617	0.3812	0.3778	0.0542
0.5	0.1173	0.0902	0.0587	0.0807	0.0898	0.4793	0.045	0.0469	0.2063	0.0438	0.4836	0.0527	0.0613	0.3797	0.3785	0.0538	
0.7	0.1184	0.0906	0.0588	0.0812	0.091	0.4826	0.045	0.0474	0.2012	0.0436	0.4867	0.0528	0.0614	0.3809	0.374	0.0538	
0.9	0.1162	0.0903	0.0589	0.0805	0.0913	0.4812	0.0447	0.0475	0.2039	0.0441	0.4872	0.0527	0.0617	0.382	0.3762	0.054	
0.99	0.1177	0.0906	0.059	0.0808	0.0904	0.4827	0.0453	0.0478	0.2081	0.0447	0.4859	0.0533	0.0619	0.3841	0.3787	0.0542	
0.2	0.3	0.474	0.244	0.1248	0.1975	0.2615	0.4154	0.1239	0.1462	0.2078	0.126	0.4364	0.1991	0.218	0.3635	0.3567	0.2012
0.5	0.4653	0.2435	0.1235	0.1959	0.2606	0.4111	0.122	0.1448	0.2035	0.1253	0.4362	0.1984	0.2186	0.3605	0.3575	0.2	
0.7	0.4502	0.2437	0.1229	0.1963	0.2599	0.4135	0.1223	0.1447	0.2025	0.1237	0.4391	0.1978	0.2169	0.3607	0.3545	0.2003	
0.																	

Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

0.1	0.3	0.2164	0.127	0.0666	0.1071	0.1276	0.4288	0.051	0.0505	0.1417	0.0492	0.4338	0.059	0.0702	0.3098	0.3127	0.0603
0.5	0.1966	0.1265	0.0659	0.107	0.127	0.4307	0.0496	0.0506	0.1358	0.0482	0.4344	0.0587	0.0701	0.3088	0.3143	0.0606	
0.7	0.2062	0.1272	0.0658	0.1066	0.1279	0.4311	0.0499	0.0497	0.138	0.0478	0.4337	0.0589	0.0696	0.3072	0.3125	0.0601	
0.9	0.2192	0.1266	0.067	0.1067	0.1281	0.433	0.0506	0.0505	0.1381	0.0481	0.4331	0.0588	0.0696	0.3083	0.3141	0.0606	
0.99	0.2157	0.1272	0.0673	0.1067	0.1285	0.4314	0.0509	0.0519	0.1416	0.0495	0.4371	0.06	0.0705	0.3123	0.314	0.0615	
0.2	0.3	0.8167	0.3015	0.1318	0.2281	0.322	0.3656	0.128	0.1495	0.1644	0.1299	0.3924	0.2204	0.2441	0.3039	0.3089	0.2223
0.5	0.835	0.2981	0.128	0.2295	0.3215	0.3613	0.1273	0.1483	0.1557	0.128	0.3861	0.2198	0.2427	0.2987	0.302	0.2241	
0.7	0.834	0.2996	0.1277	0.2261	0.3211	0.3616	0.1236	0.1461	0.1534	0.1288	0.3871	0.2183	0.2414	0.2973	0.3022	0.2202	
0.9	0.8389	0.3006	0.1297	0.229	0.3251	0.362	0.1248	0.1476	0.1546	0.1285	0.3858	0.2206	0.2424	0.2984	0.3069	0.2244	
0.99	0.8086	0.3018	0.1395	0.2301	0.3215	0.3684	0.132	0.1544	0.1644	0.1347	0.3962	0.2242	0.2494	0.3094	0.3131	0.2254	
0.3	0.3	1.8673	0.4386	0.2038	0.3216	0.5201	0.3286	0.2594	0.2978	0.2182	0.2624	0.3812	0.4865	0.5227	0.3299	0.3269	0.4908
0.5	1.9424	0.4399	0.1952	0.3223	0.52	0.3221	0.2556	0.2946	0.2061	0.2519	0.3698	0.4853	0.5151	0.3197	0.3196	0.4951	
0.7	1.9201	0.438	0.1941	0.3132	0.5199	0.3141	0.25	0.2917	0.2053	0.2561	0.3696	0.4847	0.5188	0.3202	0.3214	0.491	
0.9	1.9962	0.4429	0.197	0.3167	0.5248	0.3207	0.2575	0.2871	0.2174	0.2556	0.3724	0.4945	0.5246	0.319	0.3263	0.4943	
0.99	1.8937	0.4503	0.2142	0.3328	0.5275	0.3529	0.266	0.3017	0.2212	0.267	0.3974	0.497	0.5333	0.3305	0.341	0.5041	
0.4	0.3	3.4079	0.5451	0.2827	0.3958	0.754	0.3357	0.4375	0.4882	0.2907	0.4534	0.3999	0.8646	0.9055	0.3649	0.3854	0.8758
0.5	3.3153	0.5397	0.2727	0.3851	0.7434	0.3141	0.4319	0.4806	0.2723	0.4355	0.3792	0.855	0.8931	0.3533	0.3551	0.8614	
0.7	3.8257	0.5439	0.2727	0.3905	0.7407	0.3127	0.4238	0.4744	0.2756	0.4274	0.3871	0.8515	0.8944	0.3544	0.3526	0.8849	
0.9	3.4397	0.5379	0.2812	0.39	0.7703	0.3282	0.4228	0.4811	0.2958	0.4412	0.3957	0.8557	0.3555	0.3599	0.8673		
0.99	3.2156	0.5717	0.2975	0.4119	0.759	0.3597	0.4456	0.5039	0.3077	0.4599	0.4109	0.877	0.9167	0.3909	0.3903	0.8803	
0.5	0.3	5.3211	0.662	0.3695	0.4641	1.027	0.3914	0.6669	0.7491	0.397	0.6819	0.4612	1.3531	1.4212	0.4183	0.407	0.4091
0.5	5.0768	0.6356	0.3554	0.4489	0.9969	0.3604	0.6492	0.7192	0.3585	0.6695	0.4213	1.3308	1.3808	0.4009	0.3995	0.4044	
0.7	5.2247	0.6269	0.3421	0.4431	0.9877	0.3518	0.6398	0.7126	0.3595	0.665	0.4084	1.3285	1.3754	0.412	0.3793	0.3965	
0.9	5.2629	0.6324	0.3684	0.4489	0.9989	0.3716	0.6828	0.7278	0.3765	0.6588	0.4513	1.3421	1.3827	0.4083	0.3917	0.4022	
0.99	5.2638	0.6598	0.3941	0.4734	1.0091	0.3979	0.6823	0.7545	0.4201	0.6908	0.4725	1.3497	1.393	0.4411	0.4459	0.4442	
0.6	0.3	7.7029	0.7122	0.4556	0.545	1.2966	0.4548	0.9363	1.0338	0.506	0.9663	0.4873	1.932	1.9853	0.4989	0.5003	1.9455
0.5	7.2091	0.692	0.445	0.5191	1.2782	0.4621	0.9199	0.9965	0.4673	0.9331	0.4624	1.9252	1.9685	0.4582	0.4336	1.9279	
0.7	7.7325	0.7132	0.4227	0.5138	1.2817	0.4281	0.9311	0.986	0.4658	0.9734	0.4817	1.921	1.964	0.449	0.4511	1.919	
0.9	7.3089	0.7105	0.4519	0.5105	1.2967	0.4457	0.9396	1.0035	0.4891	0.9495	0.4667	1.9164	1.9648	0.4617	0.4719	1.9359	
0.99	7.5539	0.7745	0.4828	0.5684	1.316	0.4994	0.9579	1.0523	0.5192	1.0049	0.5566	1.9604	2.001	0.5429	0.5149	1.9695	
0.7	0.3	10.342	0.7856	0.5538	0.5889	1.652	0.5658	1.3261	1.3723	0.6238	1.2813	0.568	2.6381	2.7034	0.5811	0.5557	2.6406
0.5	10.298	0.8006	0.5425	0.568	1.643	0.5993	1.2406	1.3185	0.5985	1.2571	0.5222	2.6111	2.6606	0.5295	0.5375	2.6287	
0.7	10.353	0.753	0.5098	0.5703	1.6222	0.5221	1.2358	1.3193	0.6064	1.332	0.524	2.6154	2.6564	0.5391	0.5094	2.623	
0.9	9.571	0.7888	0.5451	0.5613	1.617	0.5679	1.2359	1.3392	0.592	1.2783	0.5506	2.6011	2.678	0.5383	0.5185	2.6272	
0.99	10.108	0.8837	0.5907	0.6632	1.6528	0.635	1.2986	1.3958	0.663	1.3415	0.6035	2.6546	2.704	0.6016	0.6088	2.6596	
0.8	0.3	13.009	0.894	0.6429	0.7018	2.0723	0.7051	1.6553	1.7588	0.7782	1.6799	0.6208	3.4014	3.4801	0.645	0.624	3.4342
0.5	13.107	0.8477	0.5953	0.6649	1.992	0.65	1.5932	1.7642	0.7448	1.6278	0.6011	3.4119	3.4486	0.5931	0.5669	3.4067	
0.7	13.016	0.8031	0.6216	0.6491	1.9875	0.6641	1.6109	1.6884	0.7164	1.6121	0.6309	3.3877	3.4889	0.5722	0.5737	3.4453	
0.9	13.712	0.8363	0.6026	0.6348	2.0136	0.6982	1.6205	1.7091	0.7506	1.6398	0.6078	3.4112	3.462	0.6238	0.6131	3.4202	
0.99	13.535	1.0949	0.7063	0.7408	2.1239	0.7827	1.66	1.7828	0.8394	1.7008	0.678	3.445	3.494	0.7077	0.6778	3.5167	
0.9	0.3	16.689	0.9519	0.7742	0.7611	2.5278	0.8263	2.0526	2.1611	0.9668	2.0701	0.7108	4.3563	4.3686	0.7207	0.76	4.3633
0.5	17.19	0.9415	0.7282	0.7215	2.4479	0.8112	2.0247	2.1258	0.9019	2.0567	0.6928	4.2878	4.3697	0.6881	0.657	4.3368	
0.7	16.905	0.9344	0.701	0.6963	2.4079	0.8064	2.0149	2.1058	0.8911	2.0653	0.6385	4.2813	4.3518	0.6429	0.6528	4.2966	
0.9	16.094	0.9846	0.6879	0.7448	2.473	0.8299	2.0285	2.1263	0.917	2.0847	0.6834	4.3638	4.357	0.6833	0.7053	4.3323	
0.99	16.947	1.095	0.7967	0.8364	2.529	0.8963	2.117	2.2332	1.0456	2.1543	0.787	4.3529	4.3982	0.808	0.8156	4.3823	
1	0.3	20.58	1.119	0.847	0.8052	3.0014	1.0437	2.8393	2.6315	1.1292	2.5893	0.7983	5.3583	5.3801	0.7958	0.7969	0.844
0.5	21.216	0.9976	0.7846	0.7723	3.0111	0.9971	2.7947	2.5641	1.0883	2.5284	0.774	5.2984	5.3922	0.765	0.7517	0.7168	
0.7	21.327	1.008	0.7234	0.7755	2.9486	0.9941	2.8024	2.5519	1.0373	2.4896	0.7142	5.3231	5.3332	0.7033	0.767	0.7447	
0.9	20.063	1.1243	0.7742	0.7957	2.9658	0.9944	2.8198	2.6022	1.1143	2.5284	0.7455	5.302	5.4042	0.7587	0.8019	0.7856	
0.99	20.883	1.1552	0.8978	0.9435	2.9995	1.1022	2.9411	2.7316	1.2561	2.6378	0.8595	5.3394	5.4035	0.9358	0.9173	0.8741	

Table A.5 MSE for n=10, p=7  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and  $0.99$

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	0.4562	0.1727	0.079	0.1434	0.1721	0.3999	0.0627	0.0573	0.1003	0.0609	0.3992	0.0692	0.085	0.2489	0.2653	0.0689
0.5	0.4902	0.1708	0.0741	0.1318	0.1714	0.3946	0.0609	0.0531	0.0974	0.0578	0.397	0.065	0.0863	0.2499	0.2631	0.066	
0.7	0.4716	0.1703	0.0717	0.1343	0.1723	0.3915	0.0607	0.0549	0.0933	0.0545	0.395	0.0636	0.0749	0.2434			

Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

0.5	1.9671	0.3889	0.1506	0.2678	0.3829	0.3398	0.1383	0.1605	0.1561	0.1523	0.3611	0.2353	0.2561	0.2592	0.2815	0.2336	
0.7	2.2053	0.3627	0.1569	0.2743	0.3849	0.3306	0.1368	0.1575	0.1478	0.1443	0.4576	0.2305	0.2609	0.2576	0.273	0.2388	
0.9	1.9556	0.3567	0.1527	0.2622	0.3834	0.3397	0.1408	0.1599	0.2767	0.1494	0.3599	0.2308	0.2619	0.2661	0.2802	0.2329	
0.3	2.1304	0.3669	0.1675	0.2872	0.3897	0.3513	0.157	0.1753	0.1709	0.161	0.3807	0.2562	0.2911	0.2796	0.287	0.2404	
0.5	3.9853	0.5349	0.269	0.3791	0.6248	0.3453	0.2722	0.3197	0.2516	0.2907	0.4081	0.5345	0.5664	0.5144	0.3458	0.523	
0.7	4.5139	0.5171	0.2375	0.3938	0.6	0.3168	0.2697	0.3147	0.2447	0.2709	0.3637	0.5096	0.5477	0.306	0.3266	0.5304	
0.9	4.3579	0.5823	0.2609	0.3835	0.5857	0.3079	0.2673	0.2939	0.2383	0.2734	0.3631	0.5082	0.5443	0.3141	0.3374	0.5143	
0.3	4.3258	0.5237	0.2502	0.3758	0.734	0.3327	0.2794	0.324	0.2555	0.324	0.4248	0.5004	0.5469	0.3143	0.3495	0.5415	
0.5	7.7957	0.5446	0.316	0.4496	0.6218	0.3807	0.3248	0.3722	0.3242	0.3386	0.4271	0.5749	0.6093	0.4143	0.3852	0.6221	
0.7	7.4938	0.6979	0.3954	0.5368	0.8694	0.434	0.4863	0.5313	0.4165	0.5979	0.4294	0.9107	0.9674	0.4276	0.4331	0.9598	
0.9	7.5727	0.6538	0.352	0.4923	0.8494	0.4123	0.4622	0.5014	0.3798	0.4391	0.4557	0.9188	0.945	0.4041	0.4285	0.9139	
0.3	8.1398	0.6612	0.342	0.5277	0.833	0.3655	0.4721	0.5023	0.4209	0.448	0.4206	0.8794	0.9432	0.3723	0.4157	0.8903	
0.5	8.4947	0.6683	0.3538	0.4804	0.8171	0.3958	0.4475	0.4957	0.397	0.4902	0.4809	0.8948	0.9912	0.4462	0.4123	0.9025	
0.7	9.99	7.7357	0.723	0.4296	0.5472	0.8939	0.4735	0.5333	0.6759	0.442	0.5549	0.491	0.9595	1.0082	0.4869	0.5519	0.9626
0.9	11.844	0.8485	0.5452	0.5983	1.1761	0.5535	0.7583	0.7655	0.6097	0.7102	0.5286	1.42	1.4948	0.5551	0.5521	0.5179	
0.3	12.125	0.8284	0.5017	0.5826	1.1304	0.5274	0.6617	0.7382	0.5557	0.7714	0.4844	1.4183	1.4918	0.4975	0.4955	0.5048	
0.5	12.1	0.8007	0.4992	0.5522	1.1338	0.4912	0.6908	0.7315	0.5528	0.6868	0.5075	1.373	1.4485	0.4932	0.4954	0.4977	
0.7	11.644	0.9176	0.618	0.671	1.2149	0.6152	0.7955	0.8354	0.7099	0.7993	0.6467	1.4365	0.5125	0.5337	0.492	0.6051	
0.9	15.437	0.7776	0.5173	0.6359	1.1458	0.4991	0.7164	0.8069	0.6287	0.7048	0.5463	1.4173	1.4365	0.5125	0.5337	0.492	
0.3	16.437	0.9635	0.69	0.7752	1.4777	0.6664	1.0086	1.153	0.9236	1.0069	0.7026	2.0316	2.1027	0.6361	0.6625	2.0176	
0.5	16.186	0.9708	0.6332	1.4658	1.7296	0.6048	0.9571	1.0479	0.7578	0.9905	0.7055	2.0363	2.0623	0.7066	0.5957	1.9956	
0.7	15.922	0.9706	0.6242	0.696	1.4401	0.6446	0.9287	1.0586	0.8034	1.4082	0.6782	2.014	2.1136	0.6144	0.6037	2.0112	
0.9	16.59	1.0426	0.7133	0.7746	1.47	0.707	0.9317	1.0509	0.7934	0.9829	0.6247	1.9802	2.0471	0.5915	0.5924	2.0204	
0.3	16.962	1.1239	0.7065	0.7617	1.672	0.7728	1.0918	1.2035	0.971	1.1505	0.8332	2.2156	2.1536	0.7382	0.7632	2.1142	
0.5	24.888	1.2812	0.8275	0.9734	1.9363	0.9784	1.3478	1.4809	1.1604	1.4049	0.8416	2.6852	2.7548	0.8941	0.9086	2.8907	
0.7	22.444	1.1641	0.808	0.7748	1.7798	0.9723	1.3162	1.484	1.0269	1.3394	0.8066	2.678	2.7711	0.765	0.8727	2.7118	
0.9	26.441	1.1344	0.9058	0.8934	1.8788	0.9955	1.3518	1.6916	1.2098	1.3079	0.8513	2.7071	2.8007	0.7345	0.8166	2.8433	
0.3	21.963	1.3817	0.9892	0.9823	2.0563	1.0191	1.5962	1.5924	1.193	1.509	0.9218	2.829	2.9322	0.9858	1.0162	2.8919	
0.5	29.604	1.483	1.0522	0.994	2.4202	1.2176	1.7845	1.8686	1.4965	1.8101	0.9184	3.5889	3.6244	0.9315	1.0252	3.635	
0.7	28.554	1.4271	0.9593	0.8909	2.3426	1.1054	1.6974	1.7289	1.2995	1.739	0.8596	3.5235	3.6383	0.9668	0.9562	3.4818	
0.9	28.047	1.4816	1.0093	1.0506	2.3274	1.2286	1.5965	1.9057	1.3523	1.7127	0.9038	3.4973	3.5308	1.0633	0.8577	3.4964	
0.3	29.744	1.4471	1.1785	1.2122	2.7599	1.2334	1.7205	1.7731	1.3554	1.7175	1.0599	3.5722	3.549	1.04	0.934	3.6429	
0.5	27.623	1.5732	1.3088	1.1832	2.5508	1.6575	1.858	1.9351	1.5066	2.0759	1.2855	3.6867	3.9472	1.1554	1.1917	3.7292	
0.7	39.589	1.6693	1.2118	1.1989	2.8901	1.5682	2.0747	2.307	1.5968	2.1735	1.1953	4.6468	4.6426	1.1897	1.3111	4.5411	
0.9	36.729	1.5176	1.1925	1.0947	2.772	1.3803	2.2108	2.3116	1.7202	2.2299	1.0826	4.402	4.5078	1.0583	1.2851	4.5003	
0.3	40.434	1.4892	1.3087	1.1826	2.5897	1.3783	2.066	2.4187	1.633	2.0628	1.0283	4.4398	4.7083	1.1054	1.2461	4.3987	
0.5	38.868	1.8159	1.6965	1.1069	2.6063	1.5877	2.2056	2.1679	1.7168	2.2777	1.1917	4.4331	4.5418	1.0691	1.1668	4.6309	
0.7	47.265	2.4009	4.8902	1.4417	2.985	1.6596	2.4089	2.4607	1.97	2.4075	1.3515	5.1059	4.8839	1.5154	1.3721	4.7506	
0.9	49.825	2.039	1.3983	1.379	3.4029	1.8996	2.984	2.8655	1.9392	2.7664	1.3401	5.736	5.7129	1.5661	1.3794	1.3084	
0.3	54.344	1.7378	1.2477	1.1785	3.1209	1.6753	2.8568	2.8039	2.1848	2.6005	1.2837	5.4722	5.456	1.3343	1.255	1.3044	
0.5	43.333	1.7353	1.4149	1.2537	3.7301	1.6963	2.837	2.6135	2.1536	2.6252	1.1754	5.5102	5.5915	1.2901	1.6935	1.5999	
0.7	46.942	1.9341	1.2503	1.4071	3.2504	1.8873	3.1506	2.6469	2.054	2.6716	1.3734	5.7413	5.6843	1.3377	1.418	1.3243	
0.9	50.461	2.0485	1.7226	1.635	3.9827	2.0994	3.2536	3.0822	2.0955	3.0877	1.5177	5.707	5.867	1.8019	1.6153	1.5335	

Table A.6 MSE for n=10, p=8  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	2.9549	1.2337	0.1258	0.191	0.5006	0.4432	0.1221	0.1059	0.1308	0.1375	0.4116	0.1992	0.1087	0.2687	0.2855	0.0912
0.5	2.6784	0.2427	0.131	0.1907	0.2328	0.4003	0.1171	0.0868	0.1183	0.1157	1.1506	0.0909	0.0986	0.2179	0.3711	0.0968	
0.7	2.0988	0.2385	0.3338	0.172	0.2288	0.4091	0.123	0.1158	0.1113	0.104	0.3699	0.8754	0.103	0.2287	0.2456	0.1017	
0.9	2.1938	0.2436	0.1108	0.17	0.2528	0.4061	0.1198	0.1164	0.0963	0.0966	0.41	0.0983	0.0897	0.2367	0.2508	0.1526	
0.2	0.3	9.1348	0.5958	0.3137	0.4232	0.8736	0.43	0.7169	0.3938	0.4537	0.4499	0.6261	0.4472	0.5545	0.973	0.3943	0.3445
0.5	9.7517	0.6294	0.275	0.4858	0.5717	0.4555	0.3838	0.3659	0.332	0.3964	0.4425	0.3771	0.3832	0.3814	0.3243	0.4911	0.3445
0.7	8.9649	0.592	0.2565	0.4916	0.5425	0.4365	0.2823	0.2942	0.2831	0.2908	0.5911	0.4541	0.3499	0.3318	0.3343	0.6223	
0.9	18.353	0.7273	1.1987	0.4488	0.6188	0.4497	0.2924	0.455	0.2758	0.3574	0.5496	0.8178	0.4265	0.4928	1.4774	0.3471	
0.3	99.91	3.0117	0.3989	0.5388	0.5767	0.5461	0.5443	0.6697	0.299	0.299	0.8569	0.4566	0.5573	0.7159	0.9243	0.5233	
0.5	33.094	1.3983	2.4503	1.0505	2.0029	0.9584	0.983	1.2041	1.6402	1.061	0.9525	2.2459	2.179	3.455	0.7576	1.6329	
0.7	34.641	1.															

Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	5838.2	26,957	266.52	1349	98,364	208,3439	8,3275	13,415612	2546.1	12,587	2,7771	12,216	35,883	244,3658	35,695	111.8
	0.5	636.82	7,0297	15,269	133,72	20,024	167,5114	96,587	23097,718	177,06	10,017	20,43	31,463	8644.1	16,11775	9,6164	25,035
	0.7	1244.2	13,169	11,291	36,507	10,956	258,0987	8,8835	71,394894	427,03	88,357	318,48	141,9	303,4266	29,021	22,863	
	0.9	124332	30,609	6,4242	131,05	165,58	8784,363	1403,9	107,64977	18,943	580,97	22,692	45,44	70,352	51,30116	9,7669	298,38
	0.99	4745.8	134,39	28,006	60,681	150,44	341,0781	120,58	5,266763	36,379	504,95	12,698	6298.9	40,995	14,9891	12,543	184,5
0.2	0.3	13721	88069	82,697	48,402	3400,9	179,9229	103,66	500,97287	756,24	344,91	21,434	27,566	120,44	6,92E+01	195,4	1634,2
	0.5	9376.5	1225.5	134,92	179,45	47,778	233,6669	1111,1	1348,2631	55,693	64,859	14451	170,87	59,499	4,10E+02	995,48	287,26
	0.7	6653.1	38,017	28,133	103,04	50,311	186,724	156,6	6620,0123	59,904	59,24	173,87	975,17	57,295	2,88E+06	38,711	812,27
	0.9	42422	23,668	287,23	446,08	35,278	398,8867	12880	40,32714	352,1	46,877	32,222	166,62	4743	1,51E+03	11119	1444,8
	0.99	6406	96,362	70,788	171,81	1672,4	264,4126	78,578	347,75535	1172	1375	144,54	395,21	2494,6	2,71E+02	140,44	37640
0.3	0.3	232955	77,957	4473	153,35	2157,2	1703,073	659,77	298,2283	125125	7134	5826,3	23822	435,68	264,9336	11590	549,3
	0.5	28725	95,493	131,2	182,64	13801	30,89398	139,31	1287,1483	478	97,084	3263,2	4169	8066,5	2933,276	198,12	26478
	0.7	349899	446,33	2119,3	88,007	3963,4	8186,243	258,91	127,8003	71,855	3695,6	9583,3	210,67	51,56	667,1697	455543	3186,1
	0.9	29176	110,19	15428	170,75	221,8	166,1579	206,38	500,5931	71,082	1183,3	946,71	194,18	190,3021	4802,3	25998	
	0.99	499764	2511,8	7805	44,501	298,23	219,4963	163,37	191,1927	86,852	500,62	126,76	11437	994,35	114,72	510,54	689435
0.4	0.3	21562	288,09	62,129	11499	4833,4	602,2734	909,25	122,3161	431,17	803244	1974,8	530,67	246,66	13081,8	342,28	217,78
	0.5	6E+07	589,27	141,62	148770	1295,3	153,9267	393,2	10582,647	6279,8	110,4	4927,6	2006,8	322,97	110,9346	13670	1457,1
	0.7	33714	4394,1	791,55	963,56	347,47	374,0383	185,71	360,8319	110592	609,04	1014	3207,5	641,47	1569,462	104,31	147,16
	0.9	1E+06	982,79	14601	1350,8	1423	163649,4	314,92	422,8652	1518,6	161,42	457,22	158,54	3E+06	547,3532	38291	107,75
	0.99	645889	541,91	11917	6409,9	8772,9	543,4284	361,15	285,3514	19679	176,12	356,72	18422	382,5407	3363	2231	
0.5	0.3	21115	342,5	1995,7	105075	3831,6	1842,91	728,49	29642,197	627,61	236,73	174427	6074,5	2227,3	11632,81	15,505	636,22
	0.5	79740	6719,4	1473,8	299,98	21218	223150,1	934,16	164,199	3330,1	789,25	351923	128,51	127,38	9776,333	15,523	1351
	0.7	225774	523,64	576,53	367,82	1310	5926,862	2638,2	32011,295	397,52	82744	459,95	988,25	2490,2	4764,527	15,301	1693,2
	0.9	44469	349,12	6339,6	8922,3	689,22	297,1052	4258,1	7523,8457	57425	94,806	927,96	574,7	179,97	1896,024	15,359	360663
	0.99	22491	11873	3619,3	3E+06	3208,4	2406,357	356,04	438682,38	60001	12467	2572,3	58077	2E+06	2493,154	15,808	164,33
0.6	0.3	83692	23616	79,788	5208,4	371560	37606,53	971,54	122694,62	507,86	700,09	1947,7	4561	747,34	2771,033	19176	278775
	0.5	324073	2514,3	621	5634,7	10884	1067,645	20678	268016,47	1293,2	913,29	2462,4	478,53	849,84	995,7145	1316,7	417,97
	0.7	41595	1809,4	2086	767,92	19740	11913,3	3814,4	9106,4494	938,26	193498	843405	2512,2	1008,8	642,2396	729,18	1271,8
	0.9	835557	616,19	900,44	745,33	411,76	10898,59	1543,3	469,2837	7910	278,5	9054,9	11267	623,98	1091,687	2284,9	14062
	0.99	17426	552942	2115	1924,3	685,73	491,3757	4259,7	2454,3258	24658	340,87	1033,9	562,06	752684	642,5755	677	677,89
0.7	0.3	7E+06	21469	5521,8	731,45	753,2	169411,9	4969,3	1962,5328	5395,1	359,34	1324,6	355645	27756	20283,1	940,12	398,41
	0.5	2E+06	42197	1688,3	2597,9	2192,5	29020,5	2419,7	15234,175	649,47	1035,4	271,65	226,53	7144,3	635,7661	369,27	222,78
	0.7	49099	1409,1	467,29	3277,4	3385,8	620,1636	582,75	212,6203	461,17	1317,8	1006,3	895,04	2276,5	817,556	1956	814008
	0.9	17660	278,07	1552	15190	948,87	7237,696	607185	6583,9221	5980,2	4345,8	2275	65759	933601	10431,32	1E+06	1867,4
	0.99	198592	7256,1	2425,5	15338	190352	9032,942	608,86	213409,37	112703	356,43	21339	702,38	2830,3	1147,746	3744,3	9E+06
0.8	0.3	7E+06	25194	1223,3	6313,8	623,21	658,9721	1826,3	200913,88	323,93	864,13	18185	2030,3	10726	2763,504	2179,9	29505
	0.5	82628	25792	1331,4	3936,6	4389,3	631,1906	107126	758,5137	2951,3	978,06	624,65	1580,9	583,31	2207,981	18399	12938
	0.7	7E+06	36798	110599	449687	9404,4	132396,6	3522,1	2753,1949	334816	2858,5	905775	3903,4	3261,3	439,562	417,38	8132,5
	0.9	271510	14753	121357	7070,1	8095,9	5350,763	39888	382656,84	940,57	3345,6	6682,2	757054	7649,3	10447,52	904,95	15253
	0.99	10291	26332	1326,3	1616,5	106039	7865,697	1734,3	2614,9079	2177,4	2353,7	2911,8	3495,7	1232,4	3464847	59200	993048
0.9	0.3	70925	2602,9	55355	680,42	6374,8	1177491	1427,2	66821,963	1935,6	1428	10662	23373	149071	43614,8	2764,4	722,21
	0.5	75229	1059,1	7513,1	2979	3616,3	80394,5	1099,5	290102,36	3339	2407,6	1817,3	187235	755078	484,8229	90434	464470
	0.7	163014	34039	702,56	1071	10670	2042,558	982,23	9254,038	8794,8	498,87	4378,8	3266,8	2467,7	5559,022	3494,3	4760,7
	0.9	2E+07	2024,5	1168,3	166697	373,76	2534,098	20432	2223,019	799,52	9578,7	9562,7	1958,9	3E+07	875,5913	1299,6	2908,1
	0.99	2E+06	577,86	1035,6	4E+06	553,74	3350,579	18271	3088,522	34387	1896,7	978,67	3497,7	12130	4185,06	38658	7485,1
1	0.3	2E+06	20768	35477	121762	2355,6	1,16E+05	6012,8	8,86E+02	151939	71208	99047	2643,2	42191	4861,138	18,123	82447
	0.5	5E+06	809623	25514	70930	27340	2,96E+08	8E+06	1,04E+07	2626,1	1033,2	2187,7	2860	18043	11219,49	17,351	361,91
	0.7	659282	4667,8	921,69	773,3	74915	2,22E+03	1707,3	6,90E+04	956328	66129	427342	907,43	722,78	15190,59	17,615	3160,2
	0.9	221041	155329	168454	68172	7227,7	1,73E+03	968,77	1,32E+03	2966,2	1265,4	809,53	87900	466,68	9351,723	17,709	1304,1
	0.99	1E+06	994739	5938	15655	63145	6,40E+05	9833,8	7,77E+03	234704	790,79	71720	109012	30859	19555,15	18,768	843,03

**Table B.1 MSE values for n=80, p=3  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99**

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}</$
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Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

	0.99	0.0308	0.0301	0.0293	0.0295	0.0304	0.8359	0.03	0.0301	0.8155	0.03	0.8384	0.0303	0.0304	0.8349	0.827	0.0303
0.2	0.3	0.1232	0.1129	0.1028	0.1051	0.1203	0.8124	0.1195	0.1198	0.8058	0.1194	0.8211	0.121	0.1211	0.8194	0.8134	0.1209
	0.5	0.1231	0.1129	0.1029	0.1052	0.1204	0.8172	0.1194	0.1198	0.8091	0.1195	0.8224	0.121	0.121	0.8197	0.8136	0.1209
	0.7	0.1231	0.1128	0.1029	0.1052	0.1203	0.8147	0.1195	0.1198	0.8098	0.1194	0.8232	0.121	0.1211	0.8236	0.8144	0.121
	0.9	0.1233	0.1129	0.1029	0.1052	0.1202	0.8139	0.1195	0.1197	0.8087	0.1195	0.8238	0.121	0.1211	0.8196	0.8147	0.1211
	0.99	0.1233	0.1128	0.1028	0.1051	0.1203	0.8114	0.1195	0.1197	0.8031	0.1195	0.8156	0.1209	0.1211	0.8166	0.8093	0.121
0.3	0.3	0.2773	0.2282	0.1923	0.1993	0.2696	0.7984	0.2686	0.2689	0.7859	0.2686	0.7981	0.2721	0.272	0.7962	0.7921	0.2721
	0.5	0.2771	0.2284	0.1928	0.1996	0.2696	0.796	0.2684	0.2689	0.7895	0.2688	0.8036	0.2723	0.2721	0.8022	0.7963	0.2719
	0.7	0.2772	0.2285	0.1929	0.1998	0.2696	0.7973	0.2688	0.269	0.7931	0.2685	0.8034	0.2723	0.2721	0.8046	0.798	0.272
	0.9	0.2772	0.2283	0.1927	0.1995	0.2696	0.7958	0.2686	0.2687	0.7918	0.2688	0.8028	0.2722	0.2723	0.8037	0.7949	0.2721
	0.99	0.2771	0.2277	0.1919	0.1988	0.2696	0.7833	0.2686	0.2689	0.7786	0.2685	0.7915	0.272	0.2724	0.7902	0.7854	0.2721
0.4	0.3	0.4928	0.351	0.2764	0.289	0.4783	0.7623	0.4773	0.4774	0.7582	0.4775	0.7719	0.4837	0.4837	0.7702	0.7632	0.4834
	0.5	0.4928	0.3522	0.2786	0.2901	0.4782	0.7702	0.4772	0.4779	0.7671	0.477	0.7818	0.4836	0.484	0.7802	0.7731	0.4835
	0.7	0.4931	0.3529	0.279	0.29	0.4784	0.7734	0.4769	0.4776	0.772	0.4774	0.7831	0.4832	0.4836	0.783	0.7767	0.484
	0.9	0.4927	0.3521	0.2774	0.2898	0.4783	0.7676	0.4772	0.4778	0.768	0.4773	0.7798	0.4835	0.4835	0.7785	0.7709	0.4836
	0.99	0.4926	0.3494	0.2741	0.2868	0.4785	0.7477	0.4773	0.4776	0.7456	0.4775	0.7585	0.4832	0.4838	0.7588	0.7516	0.4834
0.5	0.3	0.7697	0.4599	0.346	0.3613	0.7343	0.7474	0.73	0.747	0.7464	0.7284	0.7458	0.7406	0.7553	0.7562	0.7394	0.7562
	0.5	0.7701	0.4632	0.3496	0.3642	0.7467	0.7427	0.7464	0.746	0.7415	0.7458	0.7538	0.7556	0.7559	0.7525	0.7517	0.7555
	0.7	0.7702	0.4649	0.3495	0.3652	0.7471	0.7472	0.7466	0.7465	0.7457	0.7571	0.7556	0.7554	0.7569	0.7562	0.7559	0.7553
	0.9	0.7708	0.4622	0.3496	0.3628	0.747	0.7398	0.7471	0.7461	0.7382	0.7459	0.7503	0.7557	0.7559	0.7487	0.751	0.7553
	0.99	0.7701	0.4545	0.3423	0.3582	0.7471	0.7412	0.7473	0.7462	0.7098	0.7459	0.7204	0.7548	0.7565	0.7118	0.7198	0.7557
0.6	0.3	1.1087	0.5399	0.3989	0.4145	1.0745	0.6938	0.1074	1.0738	0.6927	1.0734	0.7041	0.1087	0.1084	0.7053	0.7017	0.10878
	0.5	1.1093	0.5472	0.4039	0.4186	1.0753	0.7111	0.10748	1.0747	0.7105	1.074	0.7217	0.10884	0.10879	0.7203	0.7211	0.10879
	0.8	1.1093	0.5503	0.4046	0.4207	1.0752	0.7171	0.10737	1.0735	0.714	1.0737	0.7263	0.1087	0.10884	0.7257	0.7259	0.10876
	0.9	1.1089	0.5468	0.4019	0.4177	1.0752	0.7079	0.10739	1.0742	0.7068	1.0741	0.7175	0.10884	0.1089	0.7176	0.7176	0.1088
	0.99	1.1092	0.5305	0.3892	0.4068	1.0747	0.6685	0.10741	1.074	0.6694	1.0739	0.6782	0.10879	0.10879	0.6809	0.6801	0.10885
0.7	0.3	1.5078	0.5887	0.4321	0.4475	1.4625	0.6582	1.4621	1.4622	0.6562	1.4623	0.6665	1.4806	1.4801	0.6645	0.6667	1.481
	0.5	1.5083	0.6028	0.4409	0.4564	1.4629	0.6776	1.4627	1.4628	0.6776	1.4604	0.6891	1.4817	1.4814	0.6865	0.6882	1.4807
	0.7	1.509	0.606	0.4425	0.4596	1.4609	0.6843	1.4618	1.462	0.6822	1.4613	0.6951	1.4814	1.4829	0.692	0.6943	1.481
	0.9	1.5106	0.5987	0.4392	0.454	1.4608	0.6719	1.4622	1.4633	0.6725	1.4621	0.6819	1.4819	1.4812	0.6833	0.6802	1.4818
	0.99	1.5091	0.5713	0.4195	0.4362	1.4623	0.629	1.461	1.4614	0.6275	1.4616	0.637	1.4813	1.4814	0.6369	0.6369	1.4808
0.8	0.3	1.9711	0.6077	0.4512	0.4643	1.9093	0.6196	1.9097	1.9102	0.6193	1.9103	0.626	1.9338	1.9346	0.6279	0.628	1.934
	0.5	1.9707	0.6264	0.4629	0.4783	1.9094	0.644	1.9097	1.9096	0.6415	1.9095	0.6527	1.9341	1.9354	0.6508	0.6526	1.9343
	0.7	1.971	0.6328	0.4677	0.4821	1.9102	0.6503	1.9087	1.9095	0.6511	1.9091	0.6593	1.9327	1.9343	0.6585	0.6585	1.9332
	0.9	1.9716	0.6232	0.4612	0.4739	1.9089	0.6368	1.9092	1.9093	0.6372	1.9086	0.646	1.9363	1.935	0.6475	0.6466	1.9334
	0.99	1.9717	0.5802	0.4328	0.4484	1.9101	0.5844	1.9073	1.9083	0.5848	1.9098	0.5939	1.9323	1.9343	0.5942	0.5928	1.934
0.9	0.3	2.4949	0.6046	0.4556	0.4712	2.4173	0.5817	2.4152	2.4178	0.5802	2.4155	0.5901	2.4474	2.4478	0.5888	0.5903	2.4471
	0.5	2.4942	0.6275	0.4732	0.4863	2.4171	0.6088	2.4157	2.4187	0.607	2.4174	0.6163	2.4477	2.4505	0.6161	0.6157	2.4476
	0.7	2.4965	0.6337	0.4782	0.4918	2.4174	0.6158	2.4175	2.4143	0.6155	2.418	0.6253	2.4487	2.4486	0.6257	0.6264	2.4522
	0.9	2.4959	0.6231	0.469	0.4811	2.4163	0.6022	2.4155	2.4169	0.6033	2.4156	0.6083	2.4479	2.4449	0.6101	0.6087	2.4482
	0.99	2.4952	0.5706	0.4354	0.4499	2.4171	0.5449	2.4157	2.4181	0.546	2.4157	0.5535	2.4488	2.4487	0.5536	0.5507	2.452
1	0.3	3.079	0.582	0.4518	0.466	2.9833	0.5343	2.9825	2.9837	0.5456	2.9852	0.5509	3.0206	3.02	0.5527	0.5501	3.021
	0.5	3.0796	0.6176	0.474	0.4845	2.9855	0.5749	2.9821	2.9839	0.5746	2.9832	0.5798	3.0215	3.0224	0.5811	0.5841	3.0212
	0.7	3.077	0.6216	0.4787	0.491	2.9828	0.5831	2.9831	2.9826	0.5811	2.9817	0.5924	3.0241	3.02	0.5911	0.5922	3.0209
	0.9	3.0802	0.6062	0.4676	0.4808	2.983	0.5681	2.9855	2.9848	0.5666	2.9825	0.5755	3.0224	3.0205	0.5751	0.5762	3.0222
	0.99	3.0803	0.5441	0.4277	0.4413	2.9823	0.5074	2.9822	2.9799	0.5048	2.9838	0.5112	3.0222	3.0228	0.5129	0.5139	3.0207

Table B.1 MSE for n=80, p=3  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	0.0416	0.0406	0.0386	0.0392	0.041	0.7851	0.0403	0.0405	0.7658	0.0403	0.7884	0.0517	0.041	0.7817	0.7758	0.0408
	0.5	0.0416	0.0406	0.0386	0.0392	0.041	0.7838	0.0403	0.0405	0.7676	0.0403	0.7877	0.0517	0.0409	0.7837	0.7771	0.0408
	0.7	0.0416	0.0406	0.0386	0.0392	0.041	0.7854	0.0403	0.0405	0.7668	0.0403	0.7876	0.0518	0.041	0.7821	0.7778	0.0409
	0.9	0.0416	0.0406	0.0386	0.0392	0.041	0.7857	0.0403	0.0406	0.7659	0.0403	0.7877	0.0517	0.041	0.7813	0.7774	0.0408
	0.99	0.0416	0.0406	0.0386	0.0392	0.041	0.7825	0.0403	0.0406	0.7659	0.0403	0.7878	0.0517	0.0409	0.7823	0.7772	0.0408
0.2	0.3	0.1665	0.1506	0.1275	0.1334	0.1621	0.7612	0.1607	0.1613	0.7562	0.1608	0.7714	0.				

Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

0.5	4.163	0.7074	0.4754	0.489	4.0172	0.5526	4.0153	4.0154	0.5515	4.0143	0.5595	4.0797	4.0819	0.5621	0.5623	4.0798
0.7	4.1635	0.7196	0.4807	0.494	4.0139	0.5604	4.0119	4.0171	0.5593	4.0189	0.5712	4.0806	4.0814	0.5691	0.5677	4.0817
0.9	4.1604	0.6967	0.4714	0.484	4.0185	0.5441	4.0193	4.0212	0.5444	4.0173	0.553	4.0776	4.0812	0.555	0.5553	4.0784
0.99	4.1633	0.6258	0.4328	0.4453	4.0174	0.4895	4.0186	4.0179	0.4912	4.019	0.4996	4.0791	4.0779	0.4975	0.4974	4.0821

**Table B.3 MSE for n=80, p=5  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99**

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	0.0642	0.0622	0.0559	0.0582	0.063	0.7072	0.0616	0.062	0.6882	0.0616	0.7143	0.0629	0.063	0.7072	0.7021	0.0629
	0.5	0.0641	0.0622	0.0559	0.0582	0.063	0.7076	0.0616	0.062	0.687	0.0616	0.7129	0.0628	0.063	0.7091	0.6988	0.0628
	0.7	0.0642	0.0623	0.0559	0.0582	0.063	0.7082	0.0616	0.0619	0.6881	0.0616	0.7134	0.0629	0.063	0.7053	0.7036	0.0629
	0.9	0.0641	0.0622	0.0559	0.0583	0.063	0.7087	0.0616	0.062	0.688	0.0616	0.7116	0.0629	0.063	0.7067	0.6994	0.0628
	0.99	0.0641	0.0622	0.0559	0.0582	0.0631	0.7067	0.0616	0.0619	0.689	0.0616	0.7108	0.0628	0.063	0.7071	0.6994	0.0629
0.2	0.3	0.2568	0.2276	0.1632	0.1816	0.2487	0.6854	0.2459	0.2467	0.679	0.2462	0.7006	0.2512	0.2514	0.6962	0.6903	0.251
	0.5	0.2566	0.2278	0.1634	0.1817	0.2484	0.69	0.2461	0.2465	0.681	0.2464	0.7021	0.2514	0.2514	0.6989	0.6915	0.2513
	0.7	0.2564	0.2276	0.1635	0.1817	0.2485	0.6883	0.2462	0.2466	0.6821	0.2461	0.7033	0.251	0.2515	0.6981	0.6945	0.2514
	0.9	0.2569	0.2275	0.1634	0.1817	0.2488	0.6891	0.2461	0.2466	0.6793	0.2461	0.7015	0.2515	0.2514	0.698	0.6933	0.2513
	0.99	0.2566	0.2275	0.1632	0.1814	0.2487	0.6846	0.2461	0.2467	0.6744	0.2462	0.6964	0.2511	0.2515	0.6943	0.6889	0.2509
0.3	0.3	0.577	0.4439	0.2606	0.2964	0.5558	0.6693	0.5553	0.5543	0.6674	0.5537	0.6853	0.5652	0.5654	0.6817	0.6775	0.5658
	0.5	0.5772	0.4446	0.2608	0.2973	0.5568	0.6722	0.5534	0.5542	0.6676	0.5536	0.6875	0.565	0.5655	0.6865	0.6817	0.5652
	0.7	0.5775	0.4448	0.2615	0.297	0.5564	0.6732	0.5536	0.5542	0.6729	0.5531	0.6896	0.5651	0.5654	0.6885	0.6814	0.565
	0.9	0.5779	0.4444	0.2607	0.2971	0.5567	0.6724	0.5536	0.5537	0.6682	0.5532	0.6865	0.5652	0.5657	0.6866	0.6796	0.565
	0.99	0.577	0.4431	0.2597	0.2961	0.5558	0.665	0.5536	0.5542	0.6604	0.5534	0.6768	0.5653	0.5655	0.6774	0.6709	0.5655
0.4	0.3	1.0258	0.6541	0.3369	0.3783	0.9872	0.6509	0.9843	0.9849	0.648	0.9849	0.6642	1.0054	1.0051	0.6661	0.658	1.0052
	0.5	1.0271	0.6564	0.3388	0.3804	0.9874	0.6562	0.9838	0.9842	0.6555	0.9846	0.6712	1.005	1.0053	0.6724	0.6648	1.005
	0.7	1.0266	0.6567	0.34	0.381	0.9876	0.6596	0.9846	0.9845	0.6558	0.9848	0.6743	1.005	1.0054	0.6726	0.6664	1.0055
	0.9	1.0271	0.6558	0.3386	0.3799	0.9875	0.6559	0.9844	0.9847	0.6531	0.9836	0.6691	1.0059	1.0046	0.6692	0.6654	1.005
	0.99	1.0272	0.6502	0.3343	0.3759	0.9863	0.6403	0.9839	0.9853	0.6393	0.9839	0.6547	1.005	1.0053	0.6542	0.6564	1.0055
0.5	0.3	1.605	0.8166	0.3935	0.4304	1.5402	0.6278	1.5415	1.5377	0.6274	1.5374	0.6434	1.5703	1.5711	0.6429	0.6423	1.5708
	0.5	1.6036	0.8241	0.397	0.4346	1.5396	0.6367	1.5411	1.5378	0.6371	1.5379	0.6523	1.5703	1.571	0.6538	0.6505	1.5701
	0.7	1.6055	0.826	0.3987	0.4348	1.5411	0.6419	1.5405	1.5382	0.639	1.5363	0.6546	1.5716	1.5713	0.6549	0.6517	1.57
	0.9	1.6027	0.822	0.3967	0.4331	1.5405	0.6367	1.5398	1.5389	0.6342	1.5375	0.6513	1.5709	1.5719	0.6497	0.6495	1.5699
	0.99	1.6051	0.8078	0.3891	0.425	1.5403	0.6156	1.5406	1.5381	0.6145	1.538	0.6304	1.5696	1.5704	0.6285	0.6299	1.571
0.6	0.3	2.3073	0.9146	0.4323	0.4609	2.2167	0.6044	2.2148	2.2138	0.6036	2.2143	0.6165	2.2633	2.2615	0.6183	0.6188	2.2638
	0.5	2.3103	0.9285	0.4377	0.4674	2.217	0.6149	2.2133	2.2148	0.6151	2.2136	0.6296	2.2599	2.2605	0.6296	0.6305	2.2619
	0.7	2.3107	0.9327	0.4382	0.4686	2.2157	0.6189	2.2148	2.2144	0.6192	2.2133	0.6342	2.2606	2.2613	0.6346	0.6335	2.2596
	0.9	2.3102	0.9265	0.4363	0.466	2.216	0.614	2.2152	2.2142	0.6121	2.2138	0.627	2.2635	2.2634	0.626	0.6256	2.2599
	0.99	2.3097	0.9342	0.4312	0.4642	2.2157	0.6053	2.2153	2.2143	0.5876	2.2146	0.5992	2.2613	2.2605	0.6	0.6009	2.2637
0.7	0.3	3.147	0.953	0.4541	0.4774	3.0174	0.5773	3.0116	3.0131	0.5757	3.014	0.5911	3.0792	3.0803	0.5922	0.5888	3.0755
	0.5	3.1441	0.9757	0.4613	0.4861	3.0177	0.5919	3.0133	3.0146	0.5922	3.0137	0.6069	3.0775	3.0773	0.6069	0.6065	3.0768
	0.7	3.1442	0.9814	0.4649	0.4881	3.0155	0.597	3.014	3.0112	0.5975	3.0137	0.6102	3.08	3.0777	0.6096	0.6098	3.0779
	0.9	3.1434	0.9674	0.4609	0.4834	3.0153	0.5908	3.0146	3.0154	0.588	3.0155	0.6029	3.0753	3.0761	0.6039	0.6016	3.0758
	0.99	3.1457	0.9233	0.4412	0.4642	3.0153	0.5573	3.0139	3.0125	0.5575	3.0154	0.5699	3.0786	3.0775	0.5695	0.5692	3.0777
0.8	0.3	4.1097	0.939	0.4635	0.4798	3.9399	0.5508	3.9361	3.9375	0.5496	3.9328	0.563	4.0221	4.0145	0.5636	0.5619	4.0205
	0.5	4.1059	0.9718	0.4742	0.4929	3.9366	0.5673	3.9371	3.9326	0.5674	3.9377	0.5809	4.019	4.019	0.5805	0.5819	4.0208
	0.7	4.1084	0.9802	0.4769	0.4952	3.9362	0.5732	3.937	3.933	0.574	3.936	0.5864	4.0204	4.018	0.586	0.5866	4.0205
	0.9	4.1065	0.9631	0.4727	0.489	3.9373	0.5642	3.935	3.9385	0.5634	3.9392	0.5777	4.0208	4.0201	0.5765	0.578	4.0244
	0.99	4.1076	0.8948	0.4476	0.4655	3.9387	0.5266	3.9355	3.9374	0.5277	3.9341	0.5397	4.021	4.0184	0.5393	0.5372	4.0232
0.9	0.3	5.1975	0.8963	0.4622	0.4781	4.9831	0.5238	4.9799	4.9842	0.5233	4.9854	0.5357	5.0856	5.0943	0.5362	0.5365	5.0842
	0.5	5.197	0.9321	0.4763	0.4914	4.9816	0.5443	4.9784	4.9841	0.5435	4.985	0.556	5.0873	5.09	0.5556	0.5565	5.0837
	0.7	5.196	0.944	0.4811	0.495	4.9825	0.5514	4.984	4.9796	0.5491	4.9774	0.5615	5.0907	5.087	0.5612	0.562	5.0868
	0.9	5.1978	0.9265	0.471	0.487	4.9845	0.5392	4.9808	4.9831	0.54	4.9819	0.5491	5.0885	5.0523	0.5516	0.5087	5.0877
	0.99	5.1969	0.8448	0.4422	0.4583	4.9806	0.4967	4.9791	4.9814	0.4964	4.9819	0.5065	5.0853	5.0919	0.5075	0.5081	5.0889
1	0.3	6.4138	0.8296	0.4538	0.4681	6.1548	0.4983	6.1515	6.1513	0.4978	6.1498	0.5087	6.2784	6.2863	0.5095	0.5084	6.2812
	0.5	6.4132	0.8742	0.4721	0.483	6.1576	0.5177	6.1519	6.1508	0.5215	6.1503	0.5303	6.2836	6.2813	0.5305	0.5309	6.28
	0.7	6.4192	0.8862	0.477	0.4893	6.1556	0.5241	6.1548	6.1553	0.5245	6.1493	0.5359	6.284	6.28	0.5373	0.5371	6.2809
	0.9	6.4127	0.8669														

Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

	0.99	0.0527	0.0513	0.0475	0.0488	0.0519	0.7399	0.0508	0.0511	0.7233	0.0508	0.7472	0.0517	0.0518	0.742	0.7344	0.0517
0.2	0.3	0.2109	0.1888	0.1475	0.1588	0.2048	0.7228	0.203	0.2035	0.7139	0.2031	0.7306	0.2067	0.2069	0.7289	0.7229	0.2066
	0.5	0.2111	0.189	0.1476	0.1588	0.2049	0.7236	0.2029	0.2035	0.7145	0.2029	0.7344	0.2067	0.2069	0.7318	0.7236	0.2068
	0.7	0.2109	0.1889	0.1475	0.1589	0.2049	0.7233	0.2031	0.2033	0.7161	0.2031	0.7351	0.2067	0.2068	0.7318	0.7257	0.2068
	0.9	0.2108	0.189	0.1475	0.1588	0.2049	0.7248	0.203	0.2035	0.7149	0.2031	0.7329	0.2066	0.2072	0.7328	0.7254	0.2068
	0.99	0.2111	0.1887	0.1471	0.1587	0.2048	0.7194	0.203	0.2036	0.7115	0.203	0.7303	0.2066	0.2067	0.7271	0.7225	0.2067
0.3	0.3	0.4747	0.3723	0.2457	0.271	0.459	0.7026	0.4571	0.4572	0.6986	0.4566	0.7167	0.4649	0.4648	0.7138	0.7086	0.465
	0.5	0.4749	0.3726	0.2463	0.2716	0.4586	0.7061	0.4567	0.4571	0.7039	0.4565	0.7209	0.4649	0.4651	0.7191	0.7142	0.4652
	0.7	0.4748	0.373	0.2464	0.2717	0.4589	0.7073	0.4563	0.4575	0.7029	0.4566	0.7201	0.4653	0.4649	0.7204	0.7132	0.4653
	0.9	0.4745	0.3726	0.247	0.2716	0.4584	0.7069	0.4569	0.4571	0.7026	0.4564	0.7181	0.4652	0.4655	0.7182	0.7118	0.4656
	0.99	0.4748	0.3716	0.2451	0.2704	0.459	0.6977	0.4568	0.4571	0.6931	0.4564	0.7108	0.465	0.4653	0.7077	0.6989	0.4653
0.4	0.3	0.8434	0.5553	0.3266	0.3581	0.8139	0.6801	0.8123	0.8117	0.6797	0.8116	0.693	0.8274	0.8273	0.6926	0.6885	0.8266
	0.5	0.8435	0.5573	0.3282	0.3597	0.8141	0.6869	0.8113	0.8119	0.6857	0.8116	0.7013	0.8263	0.8265	0.7016	0.6955	0.8266
	0.7	0.8437	0.5581	0.3284	0.3594	0.8138	0.6887	0.8111	0.8127	0.6864	0.8116	0.7039	0.8269	0.8266	0.7024	0.6942	0.8271
	0.9	0.8439	0.5565	0.3264	0.3594	0.814	0.687	0.812	0.812	0.6846	0.8118	0.6984	0.8271	0.8269	0.6982	0.693	0.8267
	0.99	0.8435	0.552	0.3242	0.356	0.8131	0.67	0.8113	0.8124	0.67	0.8118	0.6826	0.8265	0.8266	0.681	0.6778	0.8273
0.5	0.3	1.3177	0.7029	0.3856	0.418	1.2709	0.6563	1.2695	1.2688	0.6543	1.2676	0.6711	1.2916	1.2919	0.6698	0.6707	1.2897
	0.5	1.3186	0.7082	0.3907	0.4209	1.2693	0.6679	1.2699	1.2688	0.6648	1.2684	0.6792	1.2915	1.2922	0.6791	0.6791	1.2925
	0.7	1.3181	0.7102	0.3907	0.4215	1.2702	0.6682	1.2701	1.2687	0.6672	1.2683	0.6817	1.2919	1.292	0.6836	0.6824	1.2918
	0.9	1.3185	0.7066	0.3887	0.4203	1.2696	0.6662	1.2708	1.2696	0.6632	1.2677	0.6769	1.292	1.2925	0.6771	0.6767	1.2932
	0.99	1.3189	0.6931	0.3808	0.4136	1.2711	0.6439	1.27	1.2676	0.6394	1.2686	0.6554	1.2913	0.6541	0.6542	1.2925	
0.6	0.3	1.8973	0.7958	0.4272	0.4539	1.8285	0.6286	1.8269	1.8267	0.6291	1.8251	0.6421	1.8596	1.859	0.641	0.6422	1.8612
	0.5	1.8992	0.8091	0.4334	0.46	1.8282	0.6432	1.8263	1.8267	0.6412	1.8268	0.6559	1.858	1.8597	0.6556	0.6544	1.8592
	0.7	1.8983	0.8102	0.4358	0.4624	1.8278	0.645	1.8262	1.8263	0.6462	1.8271	0.6577	1.8598	1.8612	0.6569	0.6588	1.8594
	0.9	1.901	0.8057	0.4317	0.4584	1.8279	0.6404	1.8254	1.8278	0.6369	1.8272	0.6515	1.8594	1.8602	0.652	0.6511	1.8608
	0.99	1.8968	0.7787	0.4186	0.4454	1.8284	0.6098	1.8276	1.8276	0.608	1.8253	0.6225	1.8603	1.8619	0.6218	0.6227	1.8595
0.7	0.3	2.5867	0.8398	0.452	0.4741	2.4855	0.5987	2.4847	2.4874	0.601	2.4861	0.6122	2.5347	2.5349	0.6098	0.612	2.5328
	0.5	2.5849	0.8556	0.4603	0.4823	2.4867	0.6165	2.4848	2.4859	0.6141	2.4861	0.627	2.5333	2.5291	0.6287	0.6281	2.5325
	0.7	2.585	0.8653	0.4649	0.4848	2.4886	0.6213	2.4862	2.4853	0.6204	2.4835	0.6332	2.5319	2.5326	0.6321	0.6332	2.531
	0.9	2.5846	0.8531	0.4589	0.4806	2.4897	0.6129	2.484	2.4841	0.6117	2.4838	0.6244	2.5307	2.533	0.6225	0.6234	2.5334
	0.99	2.5819	0.8104	0.4398	0.4625	2.4865	0.5767	2.4855	2.4851	0.5751	2.4856	0.5864	2.5334	2.531	0.5884	0.5871	2.5328
0.8	0.3	3.3744	0.8344	0.4627	0.4821	3.2488	0.5678	3.2476	3.2448	0.5679	3.2456	0.5807	3.300	3.3072	0.58	0.5805	3.3082
	0.5	3.3744	0.8639	0.4762	0.4941	3.2502	0.5886	3.2467	3.2478	0.588	3.247	0.6002	3.3068	3.3079	0.6003	0.601	3.3093
	0.7	3.3735	0.871	0.4798	0.4954	3.2448	0.5958	3.2457	3.2467	0.5945	3.2459	0.606	3.3046	3.3068	0.6062	0.6067	3.3051
	0.9	3.3721	0.8577	0.4727	0.4905	3.2493	0.5834	3.2446	3.2475	0.5848	3.247	0.5959	3.3066	3.3064	0.5962	0.5966	3.3072
	0.99	3.3782	0.7981	0.4459	0.4656	3.2499	0.5432	3.2491	3.2457	0.5429	3.2456	0.5522	3.3059	3.3056	0.5548	0.5543	3.305
0.9	0.3	4.2712	0.8047	0.464	0.4778	4.1085	0.5388	4.1074	4.1095	0.5395	4.1075	0.5509	4.1854	4.1895	0.5492	0.5506	4.1843
	0.5	4.2733	0.8382	0.48	0.4945	4.1118	0.5614	4.1093	4.1094	0.5611	4.1076	0.5753	4.1851	4.1811	0.5727	0.5731	4.1844
	0.7	4.271	0.8474	0.4831	0.4994	4.1073	0.5674	4.1084	4.1114	0.567	4.1122	0.5785	4.1867	4.1872	0.5788	0.5791	4.1836
	0.9	4.2678	0.8265	0.476	0.49	4.1114	0.5558	4.1108	4.1111	0.5561	4.1084	0.567	4.1895	4.1838	0.5679	0.5688	4.1847
	0.99	4.2729	0.7577	0.4428	0.458	4.1097	0.5092	4.1106	4.11	0.5084	4.1106	0.5199	4.1827	4.1847	0.5214	0.5198	4.1848
1	0.3	5.2772	0.754	0.456	0.4697	5.0732	0.5099	5.0701	5.070	0.5102	5.0738	0.5202	5.1689	5.1684	0.5199	0.5201	5.1698
	0.5	5.2757	0.7915	0.4758	0.4884	5.0714	0.5338	5.0771	5.074	0.5354	5.0739	0.5446	5.1685	5.1626	0.5452	0.5468	5.1692
	0.7	5.2753	0.8058	0.4792	0.4931	5.0757	0.5417	5.074	5.0713	0.5593	5.0714	0.5495	5.1655	5.1679	0.5512	0.5529	5.1677
	0.9	5.2719	0.7848	0.4707	0.4823	5.0746	0.5272	5.0755	5.07	0.5291	5.0719	0.5388	5.163	5.1653	0.5391	0.5369	5.1691
	0.99	5.273	0.7028	0.432	0.4453	5.0747	0.5074	5.0736	5.0727	0.4792	5.0733	0.4885	5.1681	5.1664	0.4867	0.4868	5.1672

Table B.5 MSE for n=80, p=7  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	0.076	0.0735	0.0637	0.0745	0.6775	0.0726	0.0729	0.6573	0.0726	0.6836	0.0743	0.0744	0.676	0.0707	0.0743	
	0.5	0.0758	0.0734	0.0638	0.0675	0.0745	0.6771	0.0726	0.073	0.6558	0.0726	0.6839	0.0743	0.0743	0.6763	0.0715	0.0742
	0.7	0.0759	0.0735	0.0638	0.0676	0.0744	0.6794	0.0727	0.073	0.6578	0.0726	0.6841	0.0743	0.0744	0.6799	0.0728	0.0743
	0.9	0.0759	0.0735	0.0637	0.0676	0.0745	0.6777	0.0726	0.073	0.6564	0.0727	0.6826	0.0743	0.0743	0.6775	0.0709	0.0743
0.2	0.3	0.3039	0.2666	0.1754	0.2022	0.2933	0.6586	0.2902	0.2909	0.6492	0.2902	0.6836	0.2973	0.2973	0.6637	0.2971	0.2971
	0.5	0.3038	0.2668	0.1755	0.2023	0.2934	0.6602	0.2899	0.2905	0.6511	0.2901	0.6736	0.2971	0.2972	0.6717	0.6635	0

**Table B.6 MSE for n=80, p=8  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99**

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
0.1	0.3	0.088	0.0849	0.0711	0.0769	0.0861	0.0533	0.0839	0.0842	0.6299	0.0838	0.0581	0.086	0.0862	0.6509	0.647	0.086
	0.5	0.0881	0.0849	0.0711	0.0768	0.0862	0.0533	0.0838	0.0843	0.6285	0.0839	0.0574	0.086	0.0861	0.6518	0.6457	0.086
	0.7	0.0879	0.085	0.0711	0.0769	0.0862	0.0525	0.0839	0.0843	0.6305	0.0838	0.0574	0.086	0.0861	0.6538	0.6475	0.0861
	0.9	0.088	0.0849	0.071	0.0768	0.0862	0.0527	0.0838	0.0843	0.6306	0.0839	0.0586	0.086	0.0861	0.6513	0.6446	0.0859
	0.99	0.088	0.0849	0.0711	0.0768	0.0862	0.0509	0.0838	0.0842	0.629	0.0838	0.0565	0.0861	0.0862	0.6521	0.6448	0.0861
0.2	0.3	0.3525	0.3065	0.1852	0.2205	0.3391	0.6333	0.335	0.3357	0.6214	0.3349	0.6462	0.3445	0.3439	0.6452	0.6371	0.3441
	0.5	0.3518	0.3066	0.1851	0.2207	0.339	0.6345	0.3351	0.3357	0.6264	0.3352	0.6484	0.3438	0.3439	0.6481	0.6381	0.3443
	0.7	0.3517	0.3064	0.1852	0.2207	0.3391	0.6357	0.335	0.3359	0.626	0.3351	0.6506	0.3441	0.3439	0.6473	0.6393	0.3442
	0.9	0.3519	0.3065	0.1853	0.2207	0.3391	0.6349	0.3351	0.3358	0.6254	0.3348	0.6507	0.3441	0.3441	0.6465	0.6374	0.3439
	0.99	0.3522	0.3061	0.1848	0.2204	0.3392	0.6296	0.335	0.3357	0.6214	0.3351	0.6437	0.3437	0.344	0.641	0.6373	0.3441
0.3	0.3	0.7922	0.5865	0.2781	0.3338	0.7583	0.6187	0.7533	0.7549	0.6141	0.7532	0.633	0.7742	0.7752	0.6331	0.6268	0.7737
	0.5	0.7911	0.5872	0.2782	0.3338	0.7584	0.6217	0.7532	0.7546	0.617	0.7533	0.6381	0.7746	0.7745	0.6374	0.6288	0.7734
	0.7	0.7923	0.5877	0.2777	0.3341	0.7585	0.621	0.7531	0.7542	0.6167	0.7537	0.6392	0.773	0.7733	0.635	0.6297	0.7735
	0.9	0.7914	0.5869	0.2781	0.3343	0.7583	0.6205	0.7529	0.754	0.6144	0.7538	0.637	0.7738	0.7734	0.6358	0.6278	0.7742
	0.99	0.792	0.585	0.2759	0.3324	0.7589	0.6114	0.7534	0.7547	0.6085	0.7531	0.6308	0.7738	0.773	0.6295	0.6212	0.7739
0.4	0.3	1.4073	0.8457	0.3486	0.4033	1.3456	0.6014	1.3411	1.3411	0.5989	1.3392	0.6198	1.377	1.3752	0.618	0.611	1.3766
	0.5	1.4073	0.8499	0.3499	0.4051	1.3455	0.6048	1.3389	1.3397	0.6035	1.3399	0.6245	1.375	1.3749	0.6233	0.6169	1.3753
	0.7	1.4087	0.8505	0.3501	0.4051	1.3433	0.6088	1.3393	1.3401	0.6051	1.3393	0.6247	1.3751	1.3772	0.6263	0.6207	1.375
	0.9	1.4068	0.8495	0.3488	0.4039	1.344	0.6052	1.3396	1.3399	0.6029	1.3385	0.6243	1.3763	1.3754	0.6229	0.616	1.3763
	0.99	1.4064	0.8408	0.3459	0.4002	1.3449	0.5936	1.3398	1.3394	0.5916	1.3401	0.6104	1.3755	1.3755	0.6104	0.6057	1.3768
0.5	0.3	2.1985	1.0338	0.3994	0.4427	2.0989	0.5832	2.0997	2.0944	0.4766	2.0931	0.6016	2.1506	2.151	0.6001	0.6006	2.148
	0.5	2.2009	1.0441	0.4032	0.4468	2.0988	0.5925	2.0972	2.0945	0.4962	2.0929	0.6086	2.1503	2.1521	0.6077	0.6075	2.1498
	0.7	2.2013	1.045	0.4051	0.4477	2.0976	0.5925	2.097	2.0933	0.4996	2.0942	0.6116	2.1481	2.1494	0.6085	0.6095	2.1508
	0.9	2.1976	1.0412	0.4029	0.4457	2.0965	0.5896	2.0986	2.0933	0.4899	2.0923	0.6057	2.1489	2.1494	0.6056	0.6063	2.1492
	0.99	2.2008	1.0222	0.3953	0.4395	2.0988	0.5724	2.0975	2.0936	0.4519	2.0929	0.5888	2.1511	2.1504	0.589	0.59	2.1499
0.6	0.3	3.167	1.1353	0.4333	0.4654	3.0186	0.5636	3.0139	3.0149	0.5623	3.0151	0.5808	3.095	3.0945	0.5795	0.5817	3.0955
	0.5	3.168	1.153	0.4378	0.4711	3.0185	0.5728	3.015	3.0144	0.5725	3.013	0.5907	3.093	3.0941	0.5901	0.5906	3.0957
	0.7	3.1677	1.1584	0.439	0.4712	3.0195	0.5767	3.0128	3.0168	0.576	3.0139	0.592	3.0968	3.0973	0.5912	0.5929	3.095
	0.9	3.1646	1.1507	0.439	0.4694	3.0179	0.5711	3.0139	3.0147	0.5706	3.0148	0.5894	3.0983	3.0971	0.5885	0.5885	3.0945
	0.99	3.17	1.1103	0.4256	0.4579	3.0192	0.5502	3.015	3.0175	0.5478	3.0153	0.5665	3.0945	3.0977	0.5663	0.5658	3.0948
0.7	0.3	4.3132	1.1626	0.4528	0.4746	4.1075	0.5426	4.1044	4.1046	0.5409	4.1025	0.56	4.2168	4.2161	0.559	0.5584	4.2106
	0.5	4.3081	1.1902	0.460	0.483	4.1068	0.556	4.1055	4.1029	0.5539	4.0999	0.5713	4.2188	4.2132	0.5705	0.5705	4.2098
	0.7	4.3051	1.1938	0.4608	0.4842	4.1106	0.5578	4.1079	4.105	0.5579	4.103	0.5754	4.2107	4.2131	0.5744	0.5726	4.2142
	0.9	4.312	1.1792	0.4562	0.4806	4.1078	0.5514	4.1007	4.1027	0.5512	4.1028	0.5684	4.2118	4.2095	0.5672	0.5681	4.2111
	0.99	4.3126	1.1216	0.4414	0.4652	4.1062	0.5254	4.1046	4.1009	0.525	4.1016	0.5397	4.2113	4.214	0.54	0.5409	4.2111
0.8	0.3	5.6341	1.1248	0.4581	0.4749	5.3656	0.5206	5.3565	5.3587	0.5205	5.3571	0.5367	5.5016	5.5069	0.5343	0.5355	5.5073
	0.5	5.6289	1.164	0.469	0.4866	5.3604	0.5358	5.3591	5.3604	0.5349	5.3571	0.5505	5.5021	5.5054	0.552	0.5505	5.5112
	0.7	5.6288	1.1742	0.4716	0.4891	5.3601	0.5394	5.3612	5.357	0.5388	5.3645	0.5559	5.5008	5.5017	0.554	0.556	5.5007
	0.9	5.6361	1.1533	0.4663	0.4849	5.3647	0.5332	5.3615	5.3625	0.5314	5.3551	0.5475	5.5016	5.501	0.5472	0.5485	5.502
	0.99	5.6345	1.0751	0.4451	0.4618	5.3622	0.501	5.3559	5.3577	0.4998	5.3606	0.515	5.4994	5.5048	0.5147	0.5146	5.5005
0.9	0.3	7.1254	1.0523	0.459	0.4698	6.7883	0.5004	6.7828	6.7817	0.4983	6.783	0.5124	6.9576	6.9685	0.5121	0.5117	6.9648
	0.5	7.121	1.1037	0.4698	0.4844	6.7896	0.5155	6.7807	6.7809	0.515	6.7845	0.529	6.9685	6.9647	0.5287	0.5291	6.9612
	0.7	7.1247	1.1168	0.4723	0.4861	6.7819	0.5219	6.7759	6.7841	0.5193	6.7763	0.5352	6.9632	6.967	0.5332	0.5353	6.9663
	0.9	7.1271	1.0897	0.4673	0.4814	6.7788	0.5095	6.7823	6.7784	0.5112	6.7765	0.5257	6.9606	6.9618	0.5259	0.5248	6.9689
	0.99	7.1325	0.9972	0.4382	0.4548	6.7822	0.4756	6.7798	6.7842	0.4765	6.7801	0.489	6.9666	6.9697	0.4908	0.4877	6.9629
1	0.3	8.8028	0.9731	0.4498	0.4625	8.3785	0.4765	8.3746	8.3718	0.2329	8.3692	0.489	8.5941	8.5972	0.4888	0.4888	8.6011
	0.5	8.7977	1.0203	0.4651	0.4764	8.3783	0.4956	8.3686	8.3711	0.2402	8.3725	0.5082	8.6005	8.597	0.5089	0.5094	8.6028
	0.7	8.7966	1.0409	0.4701	0.4802	8.3754	0.4988	8.3828	8.3756	0.2437	8.3729	0.5148	8.5951	8.5961	0.5136	0.5115	8.5939
	0.9	8.8008	1.0099	0.4611	0.4732	8.379	0.4914	8.3773	8.3791	0.2394	8.3754	0.5023	8.5956	8.5977	0.5033	0.5027	8.5992
	0.99	8.7971	0.9023	0.4302	0.4407	8.3783	0.4532	8.3735	8.3703	0.2244	8.3742	0.465	8.6012	8.6023	0.4643	0.4651	8.5993

**Table B.7 MSE for n=80, p=9  $\sigma = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$  and  $\rho = 0.3, 0.5, 0.7, 0.9$  and 0.99**

$\sigma$	$\rho$	$k_0$	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$	$k_7$	$k_8$	$k_9$	$k_{10}$	$k_{11}$	$k_{12}$	$k_{13}$	$k_{14}$	$k_{15}$
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Graphical Investigation of Ridge Estimators When the Eigenvalues of the Matrix  $(X^T X)$  are Skewed

	0.9	0.402	0.3467	0.1925	0.2374	0.3864	0.6116	0.3811	0.3818	0.6028	0.381	0.6269	0.3922	0.3924	0.6246	0.6174	0.3923
0.3	0.9	0.4019	0.3463	0.1918	0.2371	0.3859	0.608	0.381	0.3817	0.5989	0.3809	0.6235	0.3922	0.3921	0.6193	0.6133	0.3923
0.3	0.9032	0.6573	0.2818	0.3465	0.863	0.5953	0.8573	0.8575	0.5917	0.8568	0.6134	0.8825	0.882	0.6121	0.6058	0.8821	
0.5	0.9046	0.6583	0.2821	0.3476	0.8625	0.5998	0.8561	0.8579	0.5933	0.8564	0.6161	0.8824	0.8825	0.6146	0.6089	0.8823	
0.7	0.9038	0.6587	0.283	0.3476	0.8625	0.5972	0.8572	0.8579	0.5954	0.8566	0.6171	0.8817	0.8817	0.6156	0.609	0.8819	
0.9	0.9032	0.6578	0.2826	0.3473	0.8633	0.5975	0.857	0.8571	0.5907	0.8559	0.6162	0.8829	0.8824	0.6142	0.6077	0.8809	
0.9	0.9036	0.6557	0.2811	0.3462	0.8623	0.5928	0.8568	0.8574	0.5869	0.8558	0.609	0.8816	0.8822	0.607	0.6015	0.883	
0.4	0.3	1.608	0.9385	0.3503	0.4103	1.5309	0.5806	1.5221	1.5234	0.5784	1.5228	0.598	1.5677	1.5694	0.5993	0.5931	1.5685
0.5	1.6066	0.9436	0.3516	0.4109	1.5297	0.5854	1.5226	1.524	0.5826	1.5235	0.6042	1.5681	1.5687	0.6036	0.5962	1.5677	
0.7	1.6081	0.9445	0.3529	0.412	1.5287	0.5867	1.5216	1.5233	0.5833	1.5241	0.6043	1.5694	1.5684	0.6046	0.5978	1.5688	
0.9	1.6069	0.9418	0.3522	0.4116	1.5287	0.584	1.5221	1.5239	0.5805	1.5222	0.6038	1.5674	1.5692	0.6031	0.5958	1.5684	
0.99	1.6071	0.9332	0.3483	0.4076	1.5289	0.5756	1.522	1.5241	0.5722	1.5233	0.5914	1.5683	1.5679	0.5932	0.5863	1.5689	
0.5	0.3	2.5105	1.1371	0.4006	0.1569	2.3873	0.5648	2.3849	2.3783	0.4694	2.3792	0.5825	2.4511	2.4515	0.5819	0.583	2.4513
0.5	2.5134	1.1486	0.4038	0.1679	2.3869	0.5712	2.3859	2.3798	0.4854	2.3804	0.589	2.4505	2.4523	0.5899	0.5891	2.4505	
0.7	2.5104	1.15	0.4038	0.1681	2.387	0.5725	2.3847	2.3814	0.4909	2.3802	0.592	2.4521	2.451	0.5919	0.5922	2.4498	
0.9	2.5132	1.1459	0.4028	0.1648	2.3873	0.57	2.3855	2.3809	0.4813	2.3805	0.5881	2.4502	2.4506	0.5878	0.5893	2.4485	
0.99	2.511	1.1244	0.396	0.1515	2.3843	0.5539	2.3855	2.3804	0.447	2.3795	0.5738	2.4512	2.4505	0.573	0.5733	2.4487	
0.6	0.3	3.6133	1.2396	0.4318	0.4648	3.4361	0.5485	3.4258	3.4265	0.5464	3.4249	0.5643	3.5285	3.5302	0.5642	0.5652	3.5322
0.5	3.6154	1.258	0.4372	0.4701	3.4312	0.5555	3.4271	3.424	0.555	3.4271	0.5751	3.5296	3.5251	0.5738	0.5738	3.5267	
0.7	3.6144	1.2627	0.4381	0.4708	3.4356	0.5583	3.4271	3.4251	0.5569	3.4261	0.5779	3.5264	3.5299	0.5754	0.5765	3.5284	
0.9	3.6138	1.2532	0.4353	0.4677	3.4309	0.5515	3.427	3.4271	0.5517	3.4271	0.5713	3.5319	3.5285	0.5728	0.5729	3.5289	
0.99	3.6163	1.212	0.4254	0.4573	3.4319	0.5346	3.4281	3.4245	0.5326	3.4293	0.5528	3.5274	3.5299	0.5518	0.5527	3.5304	
0.7	0.3	4.9206	1.2563	0.4489	0.4726	4.6695	0.5288	4.6615	4.6634	0.5284	4.6638	0.545	4.8021	4.8058	0.5452	0.5455	4.8069
0.5	4.9248	1.2851	0.4568	0.4798	4.6697	0.5386	4.6627	4.6627	0.5378	4.6656	0.5592	4.8014	4.8025	0.5573	0.5569	4.8046	
0.7	4.9218	1.2949	0.4582	0.4815	4.6699	0.542	4.6627	4.665	0.542	4.6671	0.5606	4.8044	4.8047	0.5584	0.5588	4.8074	
0.9	4.9194	1.2771	0.4552	0.4778	4.6718	0.5361	4.6648	4.6687	0.5364	4.6642	0.554	4.8054	4.8031	0.553	0.5546	4.8033	
0.99	4.921	1.2174	0.4392	0.4628	4.6732	0.5129	4.664	4.6624	0.5109	4.6635	0.5291	4.8031	4.7995	0.5297	0.5288	4.8027	
0.8	0.3	6.4235	1.2149	0.456	0.4729	6.0939	0.5078	6.0921	6.0946	0.508	6.094	0.524	6.2799	6.2705	0.524	0.5241	6.2728
0.5	6.4334	1.252	0.4659	0.4829	6.0979	0.5211	6.0876	6.0905	0.5225	6.091	0.5383	6.2747	6.274	0.538	0.5382	6.274	
0.7	6.4367	1.2638	0.4689	0.4831	6.094	0.5257	6.0953	6.0955	0.5238	6.0988	0.5414	6.2689	6.2717	0.5409	0.542	6.2706	
0.9	6.4314	1.2441	0.4634	0.4802	6.1028	0.5169	6.0883	6.0883	0.5187	6.0893	0.5353	6.2767	6.2742	0.5336	0.5359	6.2716	
0.99	6.425	1.1617	0.4424	0.4593	6.0987	0.4928	6.0861	6.0884	0.49	6.091	0.5062	6.2781	6.2834	0.505	0.5062	6.2736	
0.9	0.3	8.1373	1.1337	0.4548	0.4684	7.7115	0.4901	7.711	7.7068	0.4883	7.7174	0.5025	7.941	7.945	0.5029	0.5024	7.9397
0.5	8.1338	1.1806	0.4661	0.4795	7.7182	0.5037	7.7135	7.7073	0.5019	7.7071	0.5189	7.9382	7.9484	0.5187	0.518	7.9363	
0.7	8.139	1.1938	0.4691	0.4814	7.7096	0.5084	7.7073	7.7138	0.5062	7.7213	0.5239	7.9355	7.9403	0.5223	0.5221	7.9347	
0.9	8.133	1.1707	0.4624	0.4758	7.7234	0.4997	7.7063	7.7125	0.4988	7.7058	0.5162	7.9414	7.9456	0.5153	0.5163	7.9441	
0.99	8.1332	1.0716	0.4369	0.4522	7.7239	0.4673	7.7085	7.7074	0.4667	7.7048	0.4824	7.9443	7.943	0.4812	0.4821	7.9434	
1	0.3	10.038	1.041	0.4471	0.4578	9.5183	0.469	9.5259	9.5152	0.4694	9.5186	0.4829	9.8032	9.8029	0.4818	0.4817	9.8003
0.5	10.04	1.0912	0.4615	0.4715	9.5249	0.4856	9.5189	9.5145	0.4854	9.5201	0.5	9.7957	9.8113	0.4989	0.4987	9.8036	
0.7	10.051	1.1042	0.4651	0.4751	9.5153	0.4917	9.5254	9.5164	0.4909	9.518	0.5036	9.8034	9.7934	0.5044	0.5031	9.8113	
0.9	10.042	1.0743	0.4587	0.4674	9.5295	0.4821	9.5313	9.5189	0.4813	9.5078	0.4957	9.8095	9.8025	0.4956	0.4935	9.8013	
0.99	10.043	0.9605	0.4286	0.4413	9.5213	0.4474	9.514	9.5141	0.447	9.5194	0.4576	9.8064	9.7906	0.4591	0.4594	9.8067	