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The CPT-RICCI Scalar Curvature Symmetry in Quantum Electro-Gravity

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Abstract: In this work the quantum electro-gravitational equations are derived for half spin charged particles with the help of the hydrodynamic quantum formalism. The equations show the reversing of the trace of the Ricci curvature in passing from matter to antimatter and that the CPT transformation, associated to the reversing of the trace of the Ricci curvature, leads to a more general symmetry in quantum gravity.

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Introduction

One of the intriguing problems that the theoretical physics is facing is the integration of the general relativity with the quantum mechanics. The Einstein gravitation has a fully classical ambit, the quantum mechanics mainly concerns the small atomic or sub-atomic scale, and the fundamental interactions.

Many are the unexplained aspects of the matter on cosmological scale [1]. Even if the general relativity has opened some understanding about the cosmological dynamics [2-4] the complete explanation of generation of matter and its distribution in the universe needs the integration of the cosmological physics with the quantum one. To this end the quantum gravity (QG) represents the goal of the theoretical research [5-11]. Nevertheless, difficulties arise when one attempts to apply, to the force of gravity, the standard recepy of quantum field theories [12-13].

Recently, the author has shown that by using the quantum hydrodynamic formalism is possible to achieve a non contradictory coupling of quantum equations with the gravitational one via the derivation of the impulse energy tensor [14]. The result can be easily translated into the standard quantum formalism giving rise to equations that are independent by the hydrodynamic approach and that have clear meaning and appear well defined [14].

A first outcome of the model shows that quantum effects play an important role on the gravitational kinetics of mass density at the Planck scale such as forbidding the formation of a black hole with a mass smaller than that one of the Planck [14].

Another measurable output of the theory is the detailed description of the gravitational field of antimatter. Many and discordant are the hypotheses on the gravitational features of the antimatter [15-19]. The hydrodynamic model shows that the Ricci tensor associated to an antimatter distribution has a negative sign respect to that one of the same distribution of matter. This fact is due to the negative value of the energy function for the antimatter states [20]. This fact is of paramount importance in making the CPT symmetry compatible with the matter-antimatter repulsive behavior [20].

The objective of this work is to generalize the quantum gravitational equations (QGEs)[14] to charged particles with half spin and to show that the CPT symmetry of euclidean quantum mechanics is the particular case of a more general one that comprehends the inversion of the curvature of space-time.

The work is carried out by utilizing the hydrodynamic representation of quantum mechanics where the problem is solved as a function of two real variables, $|\psi|$ and S, that lead to the standard complex wave function $\psi = |\psi| \, exp[\frac{iS}{\hbar}]$ [21-25].

The paper is organized as follows: in the first section the hydrodynamic QGEs are briefly resumed. Then

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they are generalized to half spin charged particles. Finally, the CPT symmetry is analyzed respect to the QGEs.

2. The Impulse-Energy Tensor of Quantum States Derived Via the Hydrodynamic Quantum Equations

In this section we will use the euclidean hydrodynamic representation [21] of Klein-Gordon equation (KGE) to derive the mixed energy-impulse tensor density.

The hydrodynamic form of the Klein-Gordon equation (for scalar uncharged particles)

$$\partial_{\mu}\partial^{\mu}\psi = -\frac{m^2c^2}{\hbar^2}\psi \quad , \tag{1}$$

is given by the Hamilton-Jacobi (H-J) type equation

$$g^{\mu\nu}\frac{\partial S_{(q,t)}}{\partial q^{\mu}}\frac{\partial S_{(q,t)}}{\partial q^{\nu}} - \hbar^2\frac{\partial_{\mu}\partial^{\mu}|\psi|}{|\psi|} - m^2c^2 = 0 \tag{2.a}$$

coupled to the current conservation one [21]

$$\frac{\partial}{\partial q_{\mu}}\left(|\psi|^2\frac{\partial S}{\partial q^{\mu}}\right) = m\frac{\partial J_{\mu}}{\partial q_{\mu}} = 0 \tag{2.b}$$

where

$$S = \frac{\hbar}{2i}ln[\frac{\psi}{\psi^*}] \tag{3}$$

and where

$$J_{\mu} = (c\rho, -J_i) = \frac{i\hbar}{2m}(\psi^*\frac{\partial\psi}{\partial q^{\mu}} - \psi\frac{\partial\psi^*}{\partial q^{\mu}}) \tag{4}$$

is the 4-current. Moreover, being the 4-impulse in the hydrodynamic analogy

$$p_{\mu} = (\frac{E}{c}, -p_i) = -\frac{\partial S}{\partial q^{\mu}} \tag{5}$$

it follows that the 4-current reads

$$J_{\mu} = (c\rho, -J_i) = -|\psi|^2\frac{p_{\mu}}{m} \tag{6}$$

where

$$\rho = \frac{|\psi|^2}{mc^2}\frac{\partial S}{\partial t} \tag{7}$$

Moreover, by using (5), equation (2.a) leads to

$$\frac{\partial S}{\partial q^{\mu}}\frac{\partial S}{\partial q_{\mu}} = p_{\mu}p^{\mu} = \left(\frac{E^2}{c^2} - p^2\right) = m^2c^2\left(1 - \frac{V_{qu}}{mc^2}\right) \tag{8}$$

where $p^2 = p_i p_i$ is the modulus of the spatial momentum. Generally speaking, the hydrodynamic function $E = E_{(t)}$, but for eigenstates, for which it holds $E = E_n = const$, it follows that

$$\left(\frac{E_n{}^2}{c^2} - p_n{}^2\right) = m^2c^2\left(1-\frac{V_{qu}}{mc^2}\right)$$
$$= m^2\gamma^2\ c^2\left(1-\frac{V_{qu}}{mc^2}\right) - m^2\gamma^2\ \dot{q}^2\left(1-\frac{V_{qu}}{mc^2}\right) \tag{9}$$

from where it follows that (see appendix A)

$$E_n = \pm m\gamma c^2\sqrt{1-\frac{V_{qu}}{mc^2}} \tag{10}$$

(where the minus sign stands for antiparticles) where the quantum potential reads

$$V_{qu} = -\frac{\hbar^2}{m}\frac{\partial_\mu\partial^\mu |\psi|}{|\psi|}, \tag{11}$$

and , by using (9), that

$$p_{n\mu} = \pm m\gamma\ \dot{q}_\mu\sqrt{1-\frac{V_{qu(n)}}{mc^2}} = \frac{E_n}{c^2}\dot{q}_\mu . \tag{12}$$

Thence, the Lagrangian form of the quantum hydrodynamic equation of motion (2.a) reads [14]

$$p_\mu = -\frac{\partial L}{\partial \dot{q}^\mu}, \tag{13.a}$$
$$\dot{p}_\mu = -\frac{\partial L}{\partial q^\mu} \tag{13.b}$$

where

$$L = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_i}\dot{q}_i = -p_\mu \dot{q}^\mu \tag{14.a}$$

that, for eigenstates, reads

$$L = (\pm) - \frac{mc^2}{\gamma}\sqrt{1-\frac{V_{qu}}{mc^2}} \tag{14.b}$$

where the minus accounts for antiparticles.

The motion equation can be obtained by deriving $p_{\mu(\dot{q},q)}$ from (13.a) and then inserting it into (13.b). In the quantum case, the equation of motion is also coupled, through V_{qu}, to the mass distribution $|\psi|$ of the conservation equation (2.b).

For $\hbar \to 0$ it follows that $V_{qu} \to 0$ and the classical equations of motion are recovered.

Thence, the quantum hydrodynamic motion equation for eigenstates (omitting the subscript *n*) reads

$$\frac{dp_\mu}{ds} = \pm \frac{d}{ds}\left(mcu_\mu \left(\sqrt{1 - \frac{V_{qu}}{mc^2}} \right) \right) = -\frac{\gamma}{c} \frac{\partial L}{\partial q^\mu}$$
$$= \pm mc \frac{\partial}{\partial q^\mu} \sqrt{1 - \frac{V_{qu}}{mc^2}} \tag{15}$$

where

$$u_\mu = \frac{\gamma}{c} \dot{q}_\mu , \tag{16}$$

that leads to

$$\pm mc\sqrt{1 - \frac{V_{qu}}{mc^2}} \frac{du_\mu}{ds} = (\pm) - mcu_\mu \frac{d}{ds}\left(\sqrt{1 - \frac{V_{qu}}{mc^2}} \right) \pm mc \frac{\partial}{\partial q^\mu}\left(\sqrt{1 - \frac{V_{qu}}{mc^2}} \right) = -\frac{\gamma}{c} \frac{\partial \mathsf{T}_\mu{}^\nu}{\partial q^\nu} \tag{17}$$

where, for eigenstates, the quantum energy-impulse tensor (QEIT) $\mathsf{T}_\mu{}^\nu$ reads[14]

$$\mathsf{T}_\mu{}^\nu = \left(\dot{q}_\mu \frac{\partial L}{\partial \dot{q}_\nu} - L\delta_\mu{}^\nu \right) = \pm \frac{mc^2}{\gamma} \sqrt{1 - \frac{V_{qu}}{mc^2}} \left(u_\mu u^\nu - \delta_\mu{}^\nu \right). \tag{18}$$

leading to the quantum impulse energy tensor density (QIETD) [14]

$$T_\mu{}^\nu = \dot{q}_\mu \frac{\partial \boldsymbol{L}}{\partial \dot{q}_\nu} - \boldsymbol{L}\delta_\mu{}^\nu = |\psi|^2 \left(\dot{q}_\mu \frac{\partial L}{\partial \dot{q}_\nu} - L\delta_\mu{}^\nu \right) = |\psi|^2 \, \mathsf{T}_\mu{}^\nu \tag{19}$$

where $\boldsymbol{L} = |\psi|^2 L$ is the (quantum hydrodynamic) Lagrangian density and L is the quantum hydrodynamic Lagrangian function..

It must be noted that the hydrodynamic solutions given by (17) represent an ensemble wider than the quantum one since not all the field solutions p_μ warrant the existence of the integral action function S so that the irrotational condition has to be imposed (see references [14,22]).

Equation (17) (following the method described in ref. [14]) can be used to find the eigenstates of matter wave $\psi^+{}_n$ (by considering the upper positive sign in (18)) and the antimatter eigenstates $\psi^-{}_n$ by using the lower minus sign in (18). Furthermore, being the eigenstates irrotational (see example in appendix B) and hence, all their linear superpositions, the solution of equation (17) allows to solve the quantum problem.

Since for a generic matter-antimatter superposition of states the energy is neither positive nor negative definite, the Lagrangian function as well as the QEITD must be re-write in a more general form. To this end we observe that by using (8) it follows that (see (A.6) in appendix A)

$$p_\mu = \left(-\frac{1}{c^2} \frac{\partial S}{\partial t} \right) \dot{q}_\mu , \tag{20}$$

and that

$$L=\frac{dS}{dt}=\frac{\partial S}{\partial t}+\frac{\partial S}{\partial q_i}\dot{q}_i=-p_\mu\dot{q}^\mu=c^2\left(\frac{\partial S}{\partial t}\right)^{-1}p_\mu p^\mu=\left(\frac{|\psi|^2}{mc^2}\frac{\partial S}{\partial t}\right)^{-1}p_\mu J^\mu$$
$$=\rho^{-1}p_\mu J^\mu=c^2\left(\frac{\partial S}{\partial t}\right)^{-1}\left[m^2c^2\left(1-\frac{V_{qu}}{mc^2}\right)\right] \quad (21)$$
$$=-\frac{i\hbar}{2}c^2\left(\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial t}\right)^{-1}\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q^\mu}\left(\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q_{\mu i}}\right)$$

and that

$$T_\mu^{\ \nu}=|\psi|^2c^2\left(\frac{\partial S}{\partial t}\right)^{-1}\left(p_\mu p^\nu-p_\alpha p^\alpha\delta_\mu^{\ \nu}\right)$$
$$=\left(\frac{1}{mc^2}\frac{\partial S}{\partial t}\right)^{-1}\left(J_\mu p^\nu-J_\alpha p^\alpha\delta_\mu^{\ \nu}\right) \quad (22.a)$$
$$=\frac{m|\psi|^2c^2}{\gamma}\left(\frac{1}{m\gamma c^2}\frac{\partial S}{\partial t}\right)^{-1}\left(u_\mu u^\nu-\left(1-\frac{V_{qu}}{mc^2}\right)\delta_\mu^{\ \nu}\right)$$

$$T_\mu^{\ \nu}=(m\rho)^{-1}\left(J_\mu J^\nu-J_\alpha J^\alpha\delta_\mu^{\ \nu}\right)$$
$$=m|\psi|^2c^2\left(\frac{\frac{\hbar}{2im^2c^2}\partial ln[\frac{\psi}{\psi^*}]}{\partial t}\right)^{-1} \quad (22.b)$$
$$\left(\left(\frac{\hbar}{2mc}\right)^2\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q^\mu}\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q_\nu}+\left(1-\frac{V_{qu}}{mc^2}\right)\delta_\mu^{\ \nu}\right)$$

2.1 Non-Euclidean Generalization

Since any mass distribution leads to a non-flat space, equation (2.b, 15, 22) must be expressed in a non- euclidean space and read, respectively,

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial q^\mu}\sqrt{-g}\left(g^{\mu\nu}|\psi|\frac{\partial S}{\partial q^\nu}\right)=0 \quad (23)$$

$$\frac{du_\mu}{ds}-\frac{1}{2}\frac{\partial g_{\lambda\kappa}}{\partial q^\mu}u^\lambda u^\kappa$$
$$=-u_\mu\frac{d}{ds}\left(ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right)+\frac{\partial}{\partial q^\mu}\left(ln\sqrt{1-\frac{V_{qu}}{mc^2}}\right) \quad (24)$$

$$T_{\mu\nu} = T_{\mu}{}^{\alpha} g_{\alpha\nu} = |\psi|^2 m^2 c^4 \left(\frac{\partial S}{\partial t}\right)^{-1} \left(\frac{p_\mu p_\nu}{m^2 c^2} - \left(1 - \frac{V_{qu}}{mc^2}\right) g_{\mu\nu}\right)$$

$$= \frac{m|\psi|^2 c^2}{\gamma} \left(\frac{1}{m\gamma c^2}\frac{\partial S}{\partial t}\right)^{-1} \left(u_\mu u_\nu - \left(1 - \frac{V_{qu}}{mc^2}\right) g_{\mu\nu}\right) \quad (25.a)$$

$$T_{\mu\nu} = m|\psi|^2 c^2 \left(\frac{\frac{\hbar}{2im^2c^2}\partial ln[\frac{\psi}{\psi^*}]}{\partial t}\right)^{-1}$$

$$\left(\left(\frac{\hbar}{2mc}\right)^2 \frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q^\mu}\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q^\nu} + \left(1 - \frac{V_{qu}}{mc^2}\right) g_{\mu\nu}\right) \quad (26)$$

where

$$V_{qu} = -\frac{\hbar^2}{m}\frac{1}{|\psi|\sqrt{-g}}\partial_\mu\sqrt{-g}\left(\partial^\mu|\psi|\right), \quad (27)$$

where $\frac{1}{g} = |g_{\nu\mu}| = -J_{ac}{}^2$, where J_{ac} is jacobian of the transformation of the Galilean co-ordinates to non-euclidean ones and where $g_{\nu\mu}$ is the metric tensor defined by the quantum gravitational equation [14]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R_{\alpha}{}^{\alpha} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \quad (28)$$

where the constant Λ, that warrants the principle of minimum action and the correct Einstein classical limit [14], reads

$$\Lambda = \frac{8\pi G}{c^4}|\psi|^2 L_0 = \frac{8\pi G}{c^4}\frac{m|\psi|^2 c^2}{\gamma}\left(\frac{1}{m\gamma c^2}\frac{\partial S_0}{\partial t}\right)^{-1} \quad (29)$$

where, for scalar uncharged particles (see appendix A)

$$L_0 = m^2c^4\left(\frac{\partial S_0}{\partial t}\right)^{-1} = (\pm) - \frac{mc^2}{\gamma} \quad (30)$$

where the lower minus refers to antimatter and where (A.4))

$$S_0 = lim_{\hbar\to 0}\frac{\hbar}{2i}ln[\frac{\psi}{\psi^*}] \quad (31)$$

As shown by Bialiniki-Birula et al. and by the author himself [14,22], it is noteworthy to observe that, due to the biunique relation between the quantum hydrodynamic equations (2.a-2.b) and the quantum equation (1), equations (23-24) are equivalent to the Klein-Gordon one

$$\left(\partial^\mu\psi\right)_{;\mu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\psi\right) = -\frac{m^2c^2}{\hbar^2}\psi \quad , \quad (32)$$

(where the semicolon stands for the 4-D covariant derivative) that through the QEITD (25) couples to the quantum gravity equation (1) independently by the hydrodynamic approach .

2.2 The Overall Cosmological Constant and the Classical Limit

If we re-write the QEITD as

$$T_{\mu\nu} = \left(T_{\mu\nu\ \mathrm{CL}} - \left(\frac{1}{m\gamma c^2} \frac{\partial S}{\partial t} \right)^{-1} \frac{m|\psi|^2 c^2}{\gamma} \left(1 - \frac{V_{qu}}{mc^2} \right) g_{\mu\nu} \right) \tag{33}$$

where

$$T_{\mu\nu\ \mathrm{CL}} = \left(\frac{1}{m\gamma c^2} \frac{\partial S}{\partial t} \right)^{-1} \frac{m|\psi|^2 c^2}{\gamma} u_\mu u_\nu , \tag{34}$$

is the classical limit of the QEITD (22.a) [22] , equation (28) reads

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R_\alpha{}^\alpha - (1 - \left(1 - \frac{V_{qu}}{mc^2} \right)) \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu\ \mathrm{CL}} \tag{35}$$

that in the classical limit (when it holds $V_{qu} = 0$ as well as $\sqrt{1 - \frac{V_{qu}}{mc^2}} = 1$)) leads to the equation of the general relativity without the cosmological term that is canceled by an opposite contribution of quantum origin.

3. Gravitational Quantum Equations for Scalar Charged Particles

When we consider charged particles, we have to consider the electromagnetic (EM) interaction. This can be done by introducing in the quantum gravitational equation (1) the energy-impulse tensor for the EM field $T_{\nu\mu\ f}$ and for the matter-field interaction $T_{\mu\nu\ m-f} = T_\mu{}^\alpha{}_{m-f}\, g_{\alpha\nu}$ (derived in the following section) leading to [26]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda \delta_\mu{}^\nu = \frac{8\pi G}{c^4} \left(T_{\mu\nu\ f} + T_{\mu\nu\ m-f} \right) \quad . \tag{36}$$

coupled with the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} D^\mu \left(\sqrt{-g} D_\mu \psi \right) - \frac{A_\mu \partial^\mu \sqrt{-g}}{\sqrt{-g}} = -\frac{m^2 c^2}{\hbar^2} \psi \tag{37}$$

(where $D_\mu = \partial_\mu - \frac{e}{i\hbar} A_\mu$) and with the Maxwell one

$$F^{\mu\nu}{}_{;\nu} = -4\pi J^\mu \tag{38}$$

coupled to (36) through the energy-impulse tensor density $T_{\nu\mu\ f}$ that reads [27]

$$T_{\nu\mu\ f} = \frac{1}{4\pi} \left(-F_{\mu\lambda} F_\nu{}^\lambda + \frac{1}{4} F_{\lambda\gamma} F^{\lambda\gamma} g_{\mu\nu} \right) \tag{41}$$

where $J_\mu = \frac{\hbar}{2im} (\psi^* D_\mu \psi - \psi D_\mu \psi^*)$, where [26]

$$F_{\mu\nu} = \left(A_{\nu;\mu} - A_{\mu;\nu}\right) = \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right), \tag{39}$$

and where

$$A^\mu = (\frac{\varphi}{c}, - A_i) \tag{40}$$

is the potential 4-vector,

3.1 The Qeitd for Scalar Charged Particles

As done in section 2, we derive the mixed QEITD from which we can obtain the covariant one. The euclidean quantum hydrodynamic equations of motion for charged particles (from which the mixed QEITD can be derived) corresponding to the KGE

$$D_\mu D_\mu \psi = -\frac{m^2c^2}{\hbar^2}\psi \tag{42}$$

can be obtained by applying the minimal coupling correspondence

$$-\frac{\partial S}{\partial q_\mu} = p_\mu = \pi_\mu + eA_\mu = (\frac{E}{c}, -p_i) \tag{43}$$

(where π_μ is the mechanical momentum) to the equations (2.a) so that the H-J hydrodynamic equation reads [21]

$$\left(\frac{\partial S_{(q,t)}}{\partial q^\mu} + eA^\mu\right)\left(\frac{\partial S_{(q,t)}}{\partial q_\mu} + eA_\mu\right) = m^2c^2 + \hbar^2 \frac{\partial_\mu \partial^\mu |\psi|}{|\psi|} \tag{44}$$

united to the conservation equation

$$\frac{\partial J_\mu}{\partial q_\mu} = 0 \tag{45}$$

where the 4-current J_μ reads

$$\begin{aligned} J_\mu = (c\rho, -j_i) &= \frac{\hbar}{2im}(\psi^*\left(\partial_\mu - \frac{e}{i\hbar}A_\mu\right)\psi - \psi\left(\partial_\mu + \frac{e}{i\hbar}A_\mu\right)\psi^*) \\ &= \frac{\hbar}{2im}\left[(\psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*) - \frac{2e}{i\hbar}A_\mu\psi\psi^*\right] \\ &= -\frac{|\psi|^2}{m}\left[p_\mu - eA_\mu\right] = -\frac{|\psi|^2}{m}\pi_\mu \end{aligned} \tag{46}$$

where

$$\rho = -\frac{|\psi|^2}{mc^2}\left[\frac{\partial S}{\partial t} + e\varphi\right]. \tag{47}$$

Moreover, analogously to (8,12), from (44) it follows that

$$\pi_{n_\mu} = \frac{1}{c^2}\left[-\frac{\partial S_n}{\partial t} - e\phi\right]\dot{q}_\mu = \frac{E_n - e\phi}{c^2}\dot{q}_\mu = p_{n_\mu} - eA_\mu \tag{48.a}$$

and by (A.5) (see appendix A), that

$$\pi_\mu = \frac{1}{c^2}\left[-\frac{\partial S}{\partial t} - e\phi\right]\dot{q}_\mu = \frac{E - e\phi}{c^2}\dot{q}_\mu = p_\mu - eA_\mu \tag{49}$$

that leads to

$$L = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_i}\dot{q}_i = -p_\mu \dot{q}^\mu = -\left(\frac{1}{c^2}\frac{\partial S}{\partial t} + \frac{e}{c^2}\phi\right)^{-1} p_\mu (p^\mu - eA^\mu) = \rho^{-1} p_\mu J^\mu \tag{50}$$

that, as a function of ψ and A^μ, reads

$$L = -\frac{i\hbar}{2}c^2\left(\frac{\partial \ln[\frac{\psi}{\psi^*}]}{\partial t} + \frac{2ie}{\hbar}\varphi\right)^{-1} \frac{\partial \ln[\frac{\psi}{\psi^*}]}{\partial q^\mu}\left(\frac{\partial \ln[\frac{\psi}{\psi^*}]}{\partial q_\mu} - \frac{2ie}{\hbar}A^\mu\right) \tag{51}$$

from which, by using (19,50) with the help of (16,48), it follows that

$$T_{\mu\ \ m-f}^{\ \nu} = -|\psi|^2\left(p_\mu \dot{q}^\nu - p_\alpha \dot{q}^\alpha \delta_\mu^{\ \nu}\right) = -\left(\frac{1}{mc^2}\left(\frac{\partial S}{\partial t} + e\phi\right)\right)^{-1}\left(p_\mu J^\nu - p_\alpha J^\alpha \delta_\mu^{\ \nu}\right) = |\psi|^2 m^2 c^4\left[\frac{\partial S}{\partial t} + e\phi\right]^{-1}\left(\frac{\left(\pi_\mu + eA_\mu\right)\pi^\nu}{m^2c^2} - \left(\left(1 - \frac{V_{qu}}{mc^2}\right) + \frac{eA_\alpha \pi^\alpha}{m^2c^2}\right)\delta_\mu^{\ \nu}\right) \tag{52}$$

and finally that

$$T_\mu^{\ \nu} = -(m\rho)^{-1}\left[\left(J_\mu J^\nu - J_\alpha J^\alpha \delta_\mu^{\ \nu}\right) - \frac{e|\psi|^2}{m}\left(A_\mu J^\nu - A_\alpha J^\alpha \delta_\mu^{\ \nu}\right)\right] = \frac{m|\psi|^2 c^2}{\gamma^2}\left(\frac{1}{mc^2}\left(\frac{\partial S}{\partial t} + e\phi\right)\right)\left(\left(u_\mu + \left(\sqrt{1 - \frac{V_{qu}}{mc^2}}\right)^{-1}\frac{e}{mc}A^\nu\right)u^\nu - \left(1 + \left(\sqrt{1 - \frac{V_{qu}}{mc^2}}\right)^{-1}\frac{e}{mc}A_\alpha u^\alpha\right)\delta_\mu^{\ \nu}\right) \tag{53}$$

Moreover, by using (3,43) we can express the QEITD as a function of the wave function as

$$T_{\mu\ \ m-f}^{\ \nu} = -|\psi|^2 c^2 \left[-\frac{\partial S}{\partial t} + e\varphi \right]^{-1} \left((p_\mu - eA_\mu) p^\nu - (p_\alpha - eA_\alpha) p^\alpha \delta_\mu^{\ \nu} \right)$$

$$= |\psi|^2 \frac{\hbar c^2}{2i} \left(\frac{\partial}{\partial t} ln[\frac{\psi}{\psi^*}] - \frac{2ie}{\hbar}\varphi \right)^{-1} \left(\left(\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q^\mu} - \frac{2ie}{\hbar} A_\mu \right) \frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q_\nu} - \left(\frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q^\alpha} - \frac{2ie}{\hbar} A_\alpha \right) \frac{\partial ln[\frac{\psi}{\psi^*}]}{\partial q_\alpha} \delta_\mu^{\ \nu} \right) \quad (54)$$

4. Half Spin Charged Particles

For half spin charged particles, the QGEs are given by the gravitational one (36) coupled both to the Maxwell equation (38) (through the EM energy impulse density tensor $T_{\mu\nu\ f}$) where[22]

$$J^\mu = \overline{\Psi}\gamma^\mu \Psi , \quad (55)$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma^\mu & 0 \end{pmatrix}, \quad (56)$$

where $\sigma^\mu = (\sigma_0, \sigma_i)$ are the 4-D extended Pauli matrices [22], where

$$\Psi = (\psi, \varphi) \quad (57)$$

$$\overline{\Psi} = (\varphi^*, \psi^*), \quad (58)$$

and to the Dirac equation

$$\left(i\hbar\gamma^\mu \left(\partial_\mu + \frac{ie}{\hbar} A_\mu \right) + mc \right) \Psi = 0 \quad (59)$$

(through the energy impulse density tensor (to be defined)) $T_{\mu\nu\ m-f} = T_{\mu\ \ m-f}^{\ \alpha} g_{\alpha\nu}$.

4.1 The Quantum Energy Impulse Density Tensor and the Cosmological Constant for Half Spin Particles

As shown by Bialiniki et al. [22] the components ψ and φ of the bispinor $\Psi = \begin{pmatrix} \psi \\ \varphi \end{pmatrix}$ in the Dirac equation (56), that in the Schrödinger-like form reads

$$i\hbar\partial_t \begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \left(c\gamma^0\gamma^i \bullet \left(\frac{\hbar}{i}\nabla - eA \right) + \gamma^0 mc^2 + e\phi \right) \begin{pmatrix} \psi \\ \varphi \end{pmatrix}, \quad (60)$$

are not independent given that

$$\varphi = \frac{i\hbar}{mc} \sigma_\mu \partial^\mu \psi . \quad (61)$$

Thence, as shown by Guvenis [21], by (56 or 60) for the spinor ψ it holds the equation

$$\left(\partial_{\mu}-\frac{e}{i\hbar}A_{\mu}\right)\left(\partial^{\mu}-\frac{e}{i\hbar}A^{\mu}\right)\psi=-\frac{m^{2}c^{2}}{\hbar^{2}}\psi+\frac{ec}{\hbar}\sigma_{i}B_{i}\psi \tag{62}$$

where $B=\nabla\times A$ and where it has been used the Lorentz gauge $\nabla\bullet A=0$. By using the standard hydrodynamic notation [22] we can express the spinors ψ as

$$\psi\equiv\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=|\psi|\,exp\left[i\frac{S}{\hbar}\right]\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} \tag{63}$$

where

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}=\begin{pmatrix}exp[-i\phi/2]\cos\vartheta/2\\ exp[i\phi/2]\sin\vartheta/2\end{pmatrix} \tag{64}$$

and where ϑ and α are the angles in spherical coordinates of the spin versor

$$n_{i}=(\sin\vartheta\cos\phi,\sin\vartheta\sin\phi,\cos\vartheta)=\psi^{+}\sigma_{i}\psi\,. \tag{65}$$

and the associated 4-vector

$$n_{\mu}=(|\psi|^{2},-n_{i})=\psi^{+}\sigma_{\mu}\psi \tag{66}$$

Moreover, the current

$$\begin{aligned}J_{\mu}=(c\rho,-j_{i})&=\frac{\hbar}{2im}\left(\psi^{*}\left(\partial_{\mu}-\frac{e}{i\hbar}A_{\mu}\right)\psi-\psi\left(\partial_{\mu}-\frac{e}{i\hbar}A_{\mu}\right)\psi^{*}\right)\\ &=\frac{\hbar}{2im}\left[(\psi^{*}\partial_{\mu}\psi-\psi\partial_{\mu}\psi^{*})-\frac{2e}{i\hbar}A_{\mu}\psi\psi^{*}\right]\\ &=-\frac{|\psi|^{2}}{m}\left[-\partial_{\mu}S+\frac{\hbar}{2}\cos\vartheta\partial_{\mu}\varphi-eA_{\mu}\right]\\ &=-\frac{|\psi|^{2}}{m}\left[p_{\mu}-eA_{\mu}\right]=-\frac{|\psi|^{2}}{m}\pi_{\mu}\end{aligned} \tag{67}$$

where, in agreement with the result in ref.[22]

$$p_{\mu}=-\frac{\partial S}{\partial q^{\mu}}+\frac{\hbar}{2}\cos\vartheta\frac{\partial\phi}{\partial q^{\mu}}, \tag{68}$$

is still conserved, since

$$\left(\psi^{*}\sigma_{i}B_{i}\psi-\psi\sigma_{i}B_{i}\psi^{*}\right)=0\,, \tag{69}$$

and leads to the conservation equation,

$$\frac{\partial}{\partial q_\mu}\left(|\psi|^2 \left(\frac{\partial S}{\partial q^\mu} + eA_\mu \right)\right) = 0 \tag{70}$$

that by the hydrodynamic identity (43) leads to

$$\frac{\partial}{\partial q_\mu}\left(|\psi|^2 \ \pi_\mu \right) = m\frac{\partial}{\partial q_\mu}\left(\rho \dot{q}_\mu \right) = m\frac{\partial J_\mu}{\partial q_\mu} = 0 , \tag{71}$$

with

$$\rho = -\frac{|\psi|^2}{mc^2}\left(\frac{\partial S}{\partial t} + \frac{\hbar}{2}\cos\vartheta \frac{\partial \phi}{\partial t} + e\varphi \right). \tag{72}$$

By equating the real and imaginary part of equation (60), the conservation equation (71) is obtained together with the quantum hydrodynamic Hamilton-Jacobi motion equation [21]

$$\left(\frac{\partial S_{(q,t)}}{\partial q^\mu} + eA_\mu \right)\left(\frac{\partial S_{(q,t)}}{\partial q_\mu} + eA^\mu \right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = m^2c^2\left(1 - \frac{V_{qu\,j}}{mc^2} - \frac{\left(\Sigma_i B_i \right)_j}{mc^2} \right) \tag{73}$$

where the quantum potential

$$V_{qu} = \begin{pmatrix} V_{qu_1} \\ V_{qu_2} \end{pmatrix} = -\frac{\hbar^2}{m}\left(\frac{\partial_\mu \partial^\mu |\psi|}{|\psi|} + \partial_\mu \partial^\mu \, ln\left(\frac{\chi_1}{\chi_2} \right) + \partial_\mu \, ln\left(\frac{\chi_1}{\chi_2} \right) \partial^\mu \, ln\left(\frac{\chi_1}{\chi_2} \right) \right) \tag{74}$$

contains the contribution from the spin distributions χ_1, χ_2, where the hydrodynamic spin vector Σ_i reads

$$\Sigma_i = \begin{pmatrix} \chi_1^{-1} & 0 \\ 0 & \chi_2^{-1} \end{pmatrix} \frac{e\hbar}{mc} \sigma_i \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{75}$$

and where the "mechanical moment" (see (A.5) in appendix A) reads

$$\pi_\mu = \frac{E - e\phi + \frac{\hbar}{2}\cos\vartheta \frac{\partial \varphi}{\partial t}}{c^2} \dot{q}_\mu = -\frac{\partial S}{\partial q^\mu} + \frac{\hbar}{2}\cos\vartheta \frac{\partial \varphi}{\partial q^\mu} - eA_\mu = p_\mu - eA_\mu \cdot \tag{76}$$

Moreover, by using (68,73), it follows that

$$\left(\frac{\partial S_{(q,t)}}{\partial q^\mu} + eA^\mu \right)\left(\frac{\partial S_{(q,t)}}{\partial q_\mu} + eA_\mu \right) = \pi_\mu \pi^\mu$$

$$= \left(\frac{\left(E - e\varphi + \frac{\hbar}{2}\cos\vartheta \frac{\partial \phi}{\partial t} \right)^2}{c^2} - \left(p_i - eA_i \right)^2 \right) = m^2c^2\left(1 - \frac{V_{qu\,j}}{mc^2} - \frac{\left(\Sigma_i B_i \right)_j}{mc^2} \right) \tag{77}$$

that summing over the index j=1,2 leads to

$$\left(\frac{\partial S_{(q,t)}}{\partial q^{\mu}}+eA^{\mu}\right)\left(\frac{\partial S_{(q,t)}}{\partial q_{\mu}}+eA_{\mu}\right)=\pi_{\mu}\pi^{\mu}$$

$$=\left(\frac{\left(E-e\phi+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}\right)^{2}}{c^{2}}-\left(p_{i}-eA_{i}\right)^{2}\right)=m^{2}c^{2}\left(1-\frac{\overline{V_{qu}}}{mc^{2}}-\frac{\overline{\Sigma_{i}B_{i}}}{mc^{2}}\right) \quad (78)$$

(where $\overline{V_{qu}}=\frac{V_{qu\,1}+V_{qu\,2}}{2}$ $\overline{\Sigma_{i}B_{i}}=\frac{\left(\Sigma_{i}B_{i}\right)_{1}+\left(\Sigma_{i}B_{i}\right)_{2}}{2}$) and, hence, for eigenstates it follows that
(9)

$$E_{n}-e\phi+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}=\pm m\gamma c^{2}\sqrt{\left(1-\frac{\overline{V_{qu}}}{mc^{2}}-\frac{\overline{\Sigma_{i}B_{i}}}{mc^{2}}\right)} \quad (79)$$

that gives (for matter or antimatter eigenstates (see appendix A))

$$p_{n_{\mu}}=\left(\pm m\gamma\,\dot{q}_{\mu}\sqrt{1-\frac{\overline{V_{qu}}}{mc^{2}}-\frac{\overline{\Sigma_{i}B_{i}}}{mc^{2}}}+eA_{\mu}\right), \quad (80)$$

Moreover, by using (76) it follows that

$$L=-p_{\mu}\dot{q}^{\mu}=-\left(\frac{\frac{\partial S}{\partial t}-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}+e\phi}{c^{2}}\right)^{-1}p_{\mu}\pi^{\mu}$$

$$=\rho^{-1}p_{\mu}J^{\mu} \quad (81)$$

from which by using (5), and the identity

$$S=\frac{\hbar}{4i}ln\frac{\psi_{1}}{\psi_{1}*}\frac{\psi_{2}}{\psi_{2}*}, \quad (82)$$

the QEITD, as a function of the wave function, reads

$$T_{\mu\ \ m-f}^{\ \nu}=|\psi|^{2}c^{2}\left[\frac{\partial S}{\partial t}+e\phi+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}\right]^{-1}\left(p_{\mu}\left(p^{\nu}-eA^{\nu}\right)-p_{\alpha}\left(p^{\alpha}-eA^{\alpha}\right)\delta_{\mu}^{\ \nu}\right)$$

$$=-\left(\frac{1}{mc^{2}}\left(\frac{\partial S}{\partial t}+e\phi+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}\right)\right)^{-1}\left(p_{\mu}J^{\nu}-p_{\alpha}J^{\alpha}\delta_{\mu}^{\ \nu}\right) \quad (83)$$

$$=-(m\rho)^{-1}\left[\left(J_{\mu}J^{\nu}-J_{\alpha}J^{\alpha}\delta_{\mu}^{\ \nu}\right)-\frac{e|\psi|^{2}}{m}\left(A_{\mu}J^{\nu}-A_{\alpha}J^{\alpha}\delta_{\mu}^{\ \nu}\right)\right]$$

$$T_{\mu\;\;m-f}^{\;\nu} = |\psi|^2 \frac{4ic^2}{\hbar}\left(\frac{\partial}{\partial t} ln \frac{\psi_1}{\psi_1 *}\frac{\psi_2}{\psi_2 *} + \frac{4ie}{\hbar}\varphi + 2i\cos\vartheta \frac{\partial \phi}{\partial t}\right)^{-1} \left(\left(\frac{\partial\, ln \frac{\psi_1}{\psi_1 *}\frac{\psi_2}{\psi_2 *}}{\partial q^{\mu}} - \frac{4ie}{\hbar} A_{\mu} \right) \frac{\partial\, ln \frac{\psi_1}{\psi_1 *}\frac{\psi_2}{\psi_2 *}}{\partial q_{\nu}} - \left(\frac{\partial\, ln \frac{\psi_1}{\psi_1 *}\frac{\psi_2}{\psi_2 *}}{\partial q^{\alpha}} - \frac{4ie}{\hbar} A_{\alpha} \right) \frac{\partial\, ln \frac{\psi_1}{\psi_1 *}\frac{\psi_2}{\psi_2 *}}{\partial q_{\alpha}} \delta_{\mu}^{\;\nu} \right) \tag{84}$$

Finally, by using (81), the CC reads

$$\Lambda = -\frac{8\pi G}{c^4}|\psi|^2 L_0 = \frac{8\pi G}{c^4}|\psi|^2 \left(\frac{\frac{\partial S_0}{\partial t} + e\phi}{c^2} \right)^{-1} \frac{\partial S_0}{\partial q^{\mu}} \left(\frac{\partial S_0}{\partial q_{\mu}} - eA^{\mu} \right) \tag{85}$$

that for classical matter or antimatter states reads

$$\begin{aligned}\Lambda &= -\frac{8\pi G}{c^4}|\psi|^2 \left(\pm \frac{mc^2}{\gamma}\sqrt{1 - \frac{\overline{\Sigma_i B_i}}{mc^2}} + eA_{\mu}\dot{q}^{\mu} \right) \\ &= -\frac{8\pi G}{c^4}|\psi|^2 \left(E_0 - e\varphi + eA_{\mu}\dot{q}^{\mu} \right)\end{aligned} \tag{86}$$

where the lower minus refers to antimatter.

5. Symmetry Of The Half Spin Charged Particles QGE

The symmetry characteristic of the QGEs (36,59, 84-85) can be straightforwardly derived by observing that the Dirac equation (56) is invariant under the CPT transformation. In fact, the charge, time and parity inversion transformations lead to the substitutions

$$\begin{aligned} &C \Leftrightarrow e \rightarrow -e \\ &P \Leftrightarrow z \rightarrow -z \\ &T \Leftrightarrow t \rightarrow -t \end{aligned} \tag{87}$$

and hence to

$$CPT \Leftrightarrow \partial_{\mu} \rightarrow -\partial^{\alpha} P_{\alpha}^{\;\mu} \tag{88}$$

$$CPT \Leftrightarrow A_{\mu} \rightarrow -A^{\alpha} P_{\alpha}^{\;\mu} \tag{89}$$

$$CPT \Leftrightarrow B_i \rightarrow -B_k P_{ki} \tag{90}$$

$$CPT \Leftrightarrow \Sigma_i \rightarrow \Sigma_k P_{ki} \tag{91}$$

where $P_\alpha{}^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $P_{ki} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, that, due to the Hermitian property of $\sigma_i B_i$, applied to (62) leads to

$$CPT\left\{\left(\partial_\mu - \frac{e}{i\hbar}A_\mu\right)\left(\partial^\mu - \frac{e}{i\hbar}A^\mu\right)\psi = \left(-\frac{m^2c^2}{\hbar^2} + \frac{ec}{\hbar}\sigma_i B_i\right)\psi\right\}$$
$$= \left(\partial^\mu + \frac{e}{i\hbar}A^\mu\right)\left(\partial_\mu + \frac{e}{i\hbar}A_\mu\right)CPT(\psi) = \left(-\frac{m^2c^2}{\hbar^2} + \frac{ec}{\hbar}\sigma_i B_i\right)CPT(\psi) \qquad (92)$$

that, being equal to the Dirac equation for the complex conjugated wave function, leads to

$$CPT(\psi) = \psi^* \qquad (93)$$

from where we can see that the CPT leads a particle in its antiparticle and vice versa.
Moreover, given that $P(\vartheta) = \pi - \vartheta$, on the basis of the above identities we can apply the CPT on the QEITD to obtain

$$CPT\left(T_\mu{}^\nu{}_{m-f}\right) = T_\mu{}^\nu{}_{m-f}(\psi^*) = |\psi|^2 \frac{4ic^2}{\hbar}\left(-\frac{\partial}{\partial t} ln \frac{\psi_1{}^*}{\psi_1}\frac{\psi_2{}^*}{\psi_2} - \frac{4ie}{\hbar}\varphi - 2i\cos\vartheta\frac{\partial\phi}{\partial t}\right)^{-1}$$
$$\left(\left(\frac{\partial ln \frac{\psi_1}{\psi_1{}^*}\frac{\psi_2}{\psi_2{}^*}}{\partial q^\mu} - \frac{4ie}{\hbar}A_\mu\right)\frac{\partial ln \frac{\psi_1}{\psi_1{}^*}\frac{\psi_2}{\psi_2{}^*}}{\partial q_\nu} - \left(\frac{\partial ln \frac{\psi_1}{\psi_1{}^*}\frac{\psi_2}{\psi_2{}^*}}{\partial q^\alpha} - \frac{4ie}{\hbar}A_\alpha\right)\frac{\partial ln \frac{\psi_1}{\psi_1{}^*}\frac{\psi_2}{\psi_2{}^*}}{\partial q_\alpha}\delta_\mu{}^\nu\right) = -T_\mu{}^\nu{}_{m-f} \qquad (94)$$

Moreover, by using the time-inversion identity $T \Leftrightarrow \dot{q}^\mu \to \dot{q}_\mu$ and given that , according to (A.8) it holds

$$lim_{\hbar\to 0}\left(-\frac{\partial S_n}{\partial t} - e\phi + \frac{\hbar}{2}\cos\vartheta\frac{\partial\varphi}{\partial t}\right) = lim_{\hbar\to 0}\sqrt{\left(1 - \frac{\overline{V_{qu(n)}}}{mc^2} - \frac{\overline{\Sigma_i B_i}}{mc^2}\right)}$$
$$= \left(-\frac{\partial S_0}{\partial t} - e\phi\right) = \sqrt{\left(1 - \frac{\overline{\Sigma_i B_i}}{mc^2}\right)}$$

under CPT, it follows that

$$CPT\left[\sqrt{\left(1-\frac{\overline{\Sigma_i B_i}}{mc^2}\right)}\right]=CPT\left[\left(-\frac{\partial S_0}{\partial t}-e\phi\right)\right]$$
$$=lim_{\hbar\to 0}\,CPT\left[\left(-\frac{\partial S_n}{\partial t}-e\phi+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}\right)\right]$$
$$=lim_{\hbar\to 0}\left(\frac{\partial S_n}{\partial t}+e\phi-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}\right),$$
$$=lim_{\hbar\to 0}-\sqrt{\left(1-\frac{\overline{V_{qu\,(n)}}}{mc^2}-\frac{\overline{\Sigma_i B_i}}{mc^2}\right)}=-\sqrt{\left(1-\frac{\overline{\Sigma_i B_i}}{mc^2}\right)}$$

that

$$CPT\left(L_0\right)=CPT-\left(\pm\frac{mc^2}{\gamma}\sqrt{1-\frac{\overline{\Sigma_i B_i}}{mc^2}}+eA_\mu\dot{q}^\mu\right)$$
$$=-\left(\mp\frac{mc^2}{\gamma}\sqrt{1-\frac{\overline{\Sigma_i B_i}}{mc^2}}-eA^\mu\dot{q}_\mu\right)=-L_0 \tag{95}$$

and hence that

$$CPT\left(\Lambda\right)=CPT\left(\frac{8\pi G}{c^4}|\psi|^2\,L_0\right)=-\frac{8\pi G}{c^4}|\psi|^2\,L_0=-\Lambda. \tag{96}$$

Thence, if we consider the trace $T_\mu{}^\mu-\Lambda\delta_\mu{}^\mu=-\frac{8\pi G}{c^4}R$, where R is the trace of the Ricci curvature tensor, it follows that

$$CPT(R)=-\frac{c^4}{8\pi G}CPT\left(T_\mu{}^\mu-\Lambda\delta_\mu{}^\mu\right)=\frac{c^4}{8\pi G}\left(T_\mu{}^\mu-\Lambda\delta_\mu{}^\mu\right)=-R \tag{97}$$

given that the trace of the (antisymmetric) electromagnetic QEITD is null.
From the above result, we can infer that the CPT transformation leads to the change of the sign of the space-time scalar curvature (i.e., $R\to -R$) so that QGEs are invariant under the overall CPTR inversion transformation.

6. General Comment
On one side the quantum equation defines the evolution of the particle wave function and the associated spatial mass density distribution. On the other side, the gravity equation defines how the 4-D curvature is generated by the mass distribution and its movement through the associated tensor of energy-impulse density.

The hydrodynamic approach allows to obtain the energy-impulse tensor once the wave function of the particle is defined leading to a complete and well defined system of differential equations of evolution.

The biunique correspondence between the standard quantum mechanics and the hydrodynamic representation [22,30] warrants that once the tensor of energy-impulse density is written as a function of the wave-function, the QGEs are independent by the hydrodynamic formalism.

The CPTR symmetry of the QGEs embodies the CPT symmetry of quantum mechanics in a wider one that requires that matter and antimatter bend the space in opposite way.

This property implies interesting consequences to the quantum-gravitational dynamics.

First of all, it leads to have a repulsive matter-antimatter Galilean gravitational field [20,31]. Even if this output is the subject of discordant opinions [15], if confirmed it may bring to the elegant solution of the problem of the zero-point of quantum energy density of vacuum, an enormous amount of energy that in the gravitational approach would lead to a very high curvature of the space without matter.

If the antimatter brings a negative curvature (respect to that of matter) and since the vacuum is represented by a sea of virtual particles and antiparticles [29] (in equal number) the total curvature will be spontaneously null .
Moreover, eq. (35-36) shows the presence of the cosmological constant that, due to the quantum potential derivatives, is different from zero just in the places where the mass is localized (in quasi-punctual particles) so that the spatial mean can lead to the right order of magnitude of the cosmologically observed values [23].

Finally, the matter-antimatter repulsion can lead to their phase separation generating cosmological domains (or even universes) where matter (or antimatter) prevail [32] allowing the solution of the enigma of the abundance of matter.

7. Conclusions
In this work the quantum gravitational equations are obtained for particles interacting by means of the electromagnetic force and owning half unit of spin. This is achieved by defining the energy-impulse tensor density through the hydrodynamic quantum formalism. The electro-QGEs show to be invariant under the CPT inversion associated to the change of sign of the trace of the Ricci tensor of curvature.

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Appendix A

In this section we calculate $\frac{\partial S}{\partial t}$ and its limit $\frac{\partial S_0}{\partial t}$ for a generic superposition of states with positive (i.e., matter) and negative (antimatter) energy eigenvalues.

To this end we use the hydrodynamic expression of the wave-function

$$\psi = |\psi| \, exp[\frac{iS}{\hbar}] = \sum_n \alpha_n |\psi_n| \, exp[\frac{iS_n}{\hbar}] \qquad (A.1)$$

where

$$\psi_n = |\psi_n| \, exp[\frac{iS_n}{\hbar}] \qquad (A.2)$$

are the eigenfunctions (that for sake of simplicity we suppose with discrete eigenvalues).

For systems with time independent Hamiltonian we can write

$$\psi = |\psi| \, exp[\frac{iS}{\hbar}] = \sum_n \alpha_n |\psi_n| exp[\frac{iS_n}{\hbar}] \qquad (A.3)$$

and, hence,

$$S = \frac{\hbar}{2i} ln[\frac{\psi}{\psi^*}] = \frac{\hbar}{2i}\left(ln[\sum_n \alpha_n |\psi_n| \, exp[\frac{iS_n}{\hbar}]] - ln[\sum_n \alpha_n |\psi_n| \, exp[-\frac{iS_n}{\hbar}]] \right). \qquad (A.4)$$

By using (A.3) we obtain both

$$p_\mu = \frac{\sum_n \alpha_n \, exp[\frac{iS_n}{\hbar}] \left\{ \frac{\partial |\psi_n|}{\partial q^\mu} + \frac{i}{\hbar}\frac{\partial S_n}{\partial q^\mu} |\psi_n| \right\}}{\sum_n \alpha_n |\psi_n| \, exp[\frac{iE_n t}{\hbar}]} - \frac{\sum_n \alpha_n \, exp[\frac{iS_n}{\hbar}] \left\{ \frac{\partial |\psi_n|}{\partial q^\mu} - \frac{i}{\hbar}\frac{\partial S_n}{\partial q^\mu} |\psi_n| \right\}}{\sum_n \alpha_n |\psi_n| \, exp[-\frac{iE_n t}{\hbar}]} + \frac{\hbar}{2} cos\,\vartheta \frac{\partial \phi}{\partial q^\mu}$$
$$= -\frac{\sum_n \alpha_n |\psi_n| \, exp[\frac{iS_n}{\hbar}] \frac{\partial S_n}{\partial q^\mu}}{\sum_n \alpha_n |\psi_n| \, exp[\frac{iS_n}{\hbar}]} + \frac{\hbar}{2} cos\,\vartheta \frac{\partial \phi}{\partial q^\mu} \qquad (A.5.a)$$

$$p_{\mu}=\frac{\sum_{n}\alpha_{n}|\psi_{n}|exp[\frac{iS_{n}}{\hbar}]\left(p_{n\mu}-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\phi}{\partial q^{\mu}}\right)}{\sum_{n}\alpha_{n}|\psi_{n}|exp[\frac{iS_{n}}{\hbar}]}+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\phi}{\partial q^{\mu}}$$

$$=\frac{\sum_{n}\alpha_{n}|\psi_{n}|exp[\frac{iS_{n}}{\hbar}]\left(\pi_{n\mu}+eA_{\mu}-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\phi}{\partial q^{\mu}}\right)}{\sum_{n}\alpha_{n}|\psi_{n}|exp[\frac{iS_{n}}{\hbar}]}+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\phi}{\partial q^{\mu}} \qquad \text{(A.5.b)}$$

$$=\frac{\sum_{n}\alpha_{n}|\psi_{n}|exp[\frac{iS_{n}}{\hbar}]\left(\frac{E_{n}-e\varphi-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\phi}{\partial t}}{c^{2}}\right)}{\sum_{n}\alpha_{n}|\psi_{n}|exp[\frac{iS_{n}}{\hbar}]}\dot{q}_{\mu}+eA_{\mu}$$

$$=\left(\frac{E-e\varphi-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\phi}{\partial t}}{c^{2}}\right)\dot{q}_{\mu}+eA_{\mu}$$

$$\pi_{\mu}=p_{\mu}-eA_{\mu}=\frac{E-e\phi+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}}{c^{2}}\dot{q}_{\mu}=-\frac{\partial S}{\partial q^{\mu}}+\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial q^{\mu}}-eA_{\mu} \qquad \text{(A.6)}$$

and

$$\frac{\partial S}{\partial t}=-E=-\left(\frac{\sum_{n}\alpha_{n}|\psi_{n}|E_{n}\,exp[\frac{iS_{n}}{\hbar}]}{\sum_{n}\alpha_{n}|\psi_{n}|exp[\frac{iS_{n}}{\hbar}]}\right) \qquad \text{(A.7)}$$

$$E_{n}=\pm m\gamma c^{2}\sqrt{\left(1-\frac{\overline{V}_{qu\,(n)}}{mc^{2}}-\frac{\overline{\Sigma_{i}B_{i}}}{mc^{2}}\right)}+e\phi-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}. \qquad \text{(A.8)}$$

$$\frac{\partial S}{\partial t}=-\left(m\gamma c^{2}\frac{\sum_{n}a_{n}|\psi^{+}{}_{n}|\sqrt{\left(1-\frac{\overline{V}_{qu\,(n)}}{mc^{2}}-\frac{\overline{\Sigma_{i}B_{i}}}{mc^{2}}\right)}exp[\frac{iS_{n}t}{\hbar}]-\sum_{n}b_{n}|\psi^{-}{}_{n}|\sqrt{\left(1-\frac{\overline{V}_{qu\,(n)}}{mc^{2}}-\frac{\overline{\Sigma_{i}B_{i}}}{mc^{2}}\right)}exp[-\frac{iS_{n}t}{\hbar}]}{\sum_{n}a_{n}|\psi^{+}{}_{n}|exp[\frac{iS_{n}t}{\hbar}]+\sum_{n}b_{n}|\psi^{-}{}_{n}|exp[-\frac{iS_{n}t}{\hbar}]}+e\phi-\frac{\hbar}{2}cos\,\vartheta\frac{\partial\varphi}{\partial t}\right)$$

(A.9)

Moreover, given that for the classical limit

$$lim_{\hbar\to 0} E_n = \pm m\gamma c^2 \sqrt{\left(1-\frac{\Sigma_i B_i}{mc^2}\right)} + e\phi \qquad \forall n \tag{A.11}$$

it follows that

$$lim_{\hbar\to 0} \frac{\partial S}{\partial t} = \frac{\partial S_0}{\partial t} = -\left(m\gamma c^2 \sqrt{\left(1-\frac{\Sigma_i B_i}{mc^2}\right)} \frac{\sum_{n^+} \alpha_{n^+} exp[\frac{iS_0 t}{\hbar}] - \sum_{n^-} \alpha_{n^-} exp[-\frac{iS_0 t}{\hbar}]}{\sum_{n^+} \alpha_{n^+} exp[\frac{iS_0 t}{\hbar}] + \sum_{n^-} \alpha_{n-} exp[-\frac{iS_0 t}{\hbar}]} + e\phi \right) \tag{A.12}$$

and, hence, for pure matter or antimatter states, that

$$-\frac{\partial S_0}{\partial t} = E_0 = \pm m\gamma c^2 \sqrt{\left(1-\frac{\Sigma_i B_i}{mc^2}\right)} + e\phi \tag{A.13}$$

that for a spinless uncharged particle reads

$$-\frac{\partial S_0}{\partial t} = \pm m\gamma c^2 \tag{A.14}$$

Appendix B

Analysis of the Quantization Condition and Determination of the Quantum Eigenstates in the Quantum Hydrodynamic Description

If we look at the mathematical manageability of quantum hydrodynamic equations of quantum mechanics (2.a-2.b) no one would consider them. Nevertheless, the QHEs attract much attention by researchers. The motivation resides in the formal analogy with the classical mechanics that is appropriate to study those phenomena connecting the quantum behavior and the classical one.

In order to establish the hydrodynamic analogy, the gradient of action has to be considered as the momentum of the particle. When we do that, we broaden the solutions so that not all momenta solutions of the hydrodynamic equations can be solutions of the Schrödinger problem.

As well described in ref.[12], the state of a particle in the QHEs is defined by the real functions

$|\psi|^2 = n_{(q,\, t)}$ and $p = \nabla S_{(q,t)}$.

The restriction of the solutions of the QHEs to those ones of the standard quantum problem comes from additional conditions that must be imposed in order to obtain the quantization of the action.

The integrability of the action gradient, in order to have the scalar action function S, is warranted if the probability fluid is irrotational, that being

$$S_{(q,t)} = \int_{q_0}^{q} dl \bullet \nabla S = \int_{q_0}^{q} dl \bullet p \tag{B.1}$$

is warranted by the condition

$$\nabla \times p = 0 \tag{B.2}$$

so that it holds

$$\Gamma c = \oint dl \bullet m\dot{q} = 0 \tag{B.3}$$

Moreover, since the action is contained in the exponential argument of the wave function, all the multiples of $2\pi\hbar$, with

$$S_{n(q,t)} = S_{0(q,t)} + 2n\pi\hbar = S_{0(q_0,t)} + \int_{q_0}^{q} dl \bullet p + 2n\pi\hbar \qquad n = 0, 1, 2, 3, ... \tag{B.4}$$

are accepted.

Quantum Eigenstates

Below, we will show how the problem of finding the quantum eigenstates can be carried out in the hydrodynamic description. Since the method does not change either in classic approach or in the relativistic one, we give here an example in the simple classical case of a classical harmonic oscillator.

In the hydrodynamic description, the eigenstates are identified by their property of stationarity that is given by the "equilibrium" condition

$$\dot{p} = 0 \tag{B.5.a}$$

(that happens when the force generated by the quantum potential exactly counterbalances that one stemming from the Hamiltonian potential) with the initial "stationary" condition

$$\dot{q} = 0 \cdot \tag{B.5.b}$$

.

The initial condition (B.5.b) united to the equilibrium condition leads to the stationarity $\dot{q} = 0$ along all times and, therefore, by (B.5.a) the eigenstates are irrotational.

Since the quantum potential changes itself with the state of the system, more than one stationary state (each one with its own V_{qu_n}) is possible and more than one quantized eigenvalues of the energy may exist.

For a time independent Hamiltonian $H = \frac{p^2}{2m} + V_{(q)}$, whose hydrodynamic energy reads [31] $E = \frac{p^2}{2m} + V_{(q)} + V_{qu}$, with eigenstates $\psi_n(q)$ (for which it holds $p = m\dot{q} = 0$) it follows that

$$S_n = \int_{t_0}^{t} dt\left(\frac{p \bullet p}{2m} - V_{(q)} - V_{qu_n}\right) = -(V_{(q)} + V_{qu_n})\int_{t_0}^{t} dt = -E_n(t - t_0) \tag{B.6}$$

where $V_{qu_n} = V_{qu}(\psi_n)$, and that

$$V_{qu_n} = E_n - V_{(q)} \tag{B.7}$$

where (B.7) is the differential equation, that in the quantum hydrodynamic description, allows to derive to the eigenstates.

For instance, for a harmonic oscillator (i.e., $V_{(q)} = \frac{m\omega^2}{2} q^2$) (B.7) reads

$$V_{qu} = -(\frac{\hbar^2}{2m}) | \psi_n |^{-1} \nabla \bullet \nabla | \psi_n | = E_n - \frac{m\omega^2 q^2}{2}. \tag{B.8}$$

If for (B.8) we search a solution of type

$$| \psi |_{(q,\, t)} = A_{n(q)}\, exp\left(- a q^2\right), \tag{B.9}$$

we obtain that $a = \frac{m\omega}{2\hbar}$ and $A_{n(q)} = H_{n(\frac{m\omega}{2\hbar} q)}$ (where $H_{n(x)}$ represents the n-th Hermite polynomial). Therefore, the generic n-th eigenstate reads

$$\psi_{n(q)} = | \psi |_{(q,\, t)}\; exp\, [\, \frac{i}{\hbar} S_{(q,t)}\,] = H_{n(\frac{m\omega}{2\hbar} q)}\; exp\left(-\frac{m\omega}{2\hbar} q^2\right) exp\left(-\frac{iE_n t}{\hbar}\right), \tag{B.10}$$

From (B.10) it follows that the quantum potential of the n-th eigenstate reads

$$\begin{aligned} V_{qu}{}^{n} &= -(\frac{\hbar^2}{2m}) | \psi | \nabla_q \bullet \nabla_q | \psi | \\ &= -\frac{m\omega^2}{2} q^2 + \left[n \left(\frac{\frac{m\omega}{\hbar} H_{n-1} - 2(n-1) H_{n-2}}{H_n} \right) + \frac{1}{2} \right] \hbar\omega \\ &= -\frac{m\omega^2}{2} q^2 + (n + \frac{1}{2}) \hbar\omega \end{aligned} \tag{B.12}$$

where it has been used the recurrence formula of the Hermite polynomials

$$H_{n+1} = \frac{m\omega}{\hbar} q H_n - 2n H_{n-1}, \tag{B.13}$$

that by (B.7) leads to

$$E_n = V_{qu_n} + V_{(q)} = (n + \frac{1}{2}) \hbar\omega \tag{B.14}$$

The same result comes by the calculation of the eigenvalues that read

$$\begin{aligned} E_n = <\psi_n \mid H \mid \psi_n> &= \int_{-\infty}^{\infty} \psi^*_{(q,\,t)} H^{op} \psi_{(q,\,t)} dq \\ &= \int_{-\infty}^{\infty} |\psi|^2 \left[H_{(q,\,t)} + V_{qu}{}^{n} \right] dq \\ &= \int_{-\infty}^{\infty} \mathsf{n}_{(q,\,t)} \left[\frac{m}{2} \dot{q}^2 + \frac{m\omega^2}{2} (q - \underline{q})^2 + V_{qu}{}^{n} \right] dq \\ &= \int_{-\infty}^{\infty} \mathsf{n}_{(q,\,t)} \left[\frac{1}{2m} \nabla S_{(q)}{}^{2} + \frac{m\omega^2}{2} (q - \underline{q})^2 + V_{qu}{}^{n} \right] dq \\ &= \int_{-\infty}^{\infty} \mathsf{n}_{(q,\,t)} \left[\frac{m\omega^2}{2} (q - \underline{q})^2 - \frac{m\omega^2}{2} (q - \underline{q})^2 + (n + \frac{1}{2})\hbar\omega \right] dq = (n + \frac{1}{2})\hbar\omega \end{aligned} \quad (B.15)$$

where $H^{op} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V_{(q)}$ and where $\mathsf{n}_{(q,\,t)} = \psi^*_{(q,\,t)}\,\psi_{(q,\,t)}$. Moreover, by using (B.6, B.14-B.15) for eigenstates it follows that

$$\dot{p} = -\nabla(H + V_{qu}) = -\nabla((n + \frac{1}{2})\hbar\omega) = 0, \quad (B.16)$$

$$\dot{q} = \frac{\nabla S_{(q,t)}}{m} = 0, \quad (B.17)$$

Confirming the stationary equilibrium condition of the eigenstates.

Finally, it must be noted that since all the quantum states are given by the generic linear superposition of the eigenstates (owing the irrotational momentum field $m\dot{q} = 0$) it follows that all quantum states are irrotational. Moreover, since the Schrödinger description is complete, do not exist others quantum irrotational states in the hydrodynamic description.

In the relativistic case, the hydrodynamic solutions are determined by the eigenstates $\psi^{+}{}_{n}$, $\psi^{-}{}_{n}$ derived by the irrotational stationary equilibrium condition applied to the momentum fields of matter and antimatter of equation (23), respectively .