International Journal of Sciences

Research Article Volume 8 – March 2019 (03)

Optimization of Systems

Rowland J.O. Ekeocha[1]

[1]Covenant University, Ota, Nigeria

Abstract: Optimization is defined as the mathematical procedures involved in effecting optimality. It is also a collection of mathematical principles and methods used for solving quantitative problems in many disciplines. In business, optimization is finding an alternative with the most cost effective or highest achievable performance under given constraints by maximizing desired factors and minimizing undesired ones. The purpose of optimization is therefore to achieve the best design relative to a set of prioritized criteria or constraints which include maximizing factors such as productivity, strength, reliability, longevity, efficiency and utilization. Mathematical model attempts to optimize (maximize or minimize) an objective function without violating resource constraints. Various optimization techniques are used to achieve optimality of systems. The objective of the paper is to employed appropriate optimization techniques in worked examples to achieve optimality of systems.

Keywords: Optimality, Optimization Techniques, Systems

INTRODUCTION

Optimization is an act, process or methodology of making something such as a design, system, or decision as fully perfect, functional or effective as possible. Specifically, it is the mathematical procedures (such as finding the maximum of a function) involved in effecting optimality. (Merriam Webster).

Optimization is also a collection of mathematical principles and methods used for solving quantitative problems in many disciplines including physics, biology, engineering, economics and business. (Merriam Webster).
In mathematics and computer science, an optimization problem is one which sets out to find the best solution from all feasible solutions. Optimization problems can be divided into two categories depending on whether the variables are continuous or discrete. An optimization problem with discrete variables is known as a discrete optimization. In a discrete optimization problem, we are looking for an object such as an integer, permutation or graph from a finite or possibly countably infinite set. Problems with continuous variables include constrained problems and multimodal problems. (Wikipedia, 2018)

The classical optimization techniques are useful in finding the optimum solution or unconstrained maxima or minima of continuous and differentiable functions. These are analytical methods and make use of differential calculus in locating the optimum solution

The purpose of optimization is to achieve the best design relative to a set of prioritized criteria or constraints which include maximizing factors such as productivity, strength, reliability, longevity, efficiency and utilization. The resulting decision-making process is known as optimization. Mathematical model therefore attempts to optimize (maximize or minimize) an objective function without violating resource constraints. It also known as mathematical programming which includes Linear Programming (LP). Linear programming which is also called linear optimization is a method of achieving the best outcome such as maximum profit or lowest cost in a mathematical model whose requirements are represented by linear relationships. Linear programming is a special case of mathematical programming (mathematical optimization). (Britannica.com).

An optimization algorithm is a procedure which is executed iteratively by comparing various solutions till an optimum or a satisfactory solution is found. There are two distinct types of optimization algorithms widely used today. The first is the deterministic algorithm which uses specific rules for moving from one solution to other. The second is the stochastic algorithm which in nature is with probabilistic translation rules. It is gaining popularity due to certain properties which deterministic algorithm does not possess. (Wikipedia, 2018).

In business, optimization is finding an alternative with the most cost effective or highest achievable

Published at: http://www.ijsciences.com/pub/issue/2019-03/
DOI: 10.18483/ijSci.1964; Online ISSN: 2305-3925; Print ISSN: 2410-4477

Rowland J.O. Ekeocha (Correspondence)
+

performance under given constraints by maximizing desired factors and minimizing undesired ones. Maximization here means trying to attain the highest or maximum result or outcome without regard to cost or expenses. In computer simulation(modeling) of business problems optimization is usually achieved by the use of linear programming techniques of operation research.

Classical optimization techniques include but not limited to the following: Optimization and inequalities, numerical methods of optimization, linear programming techniques, nonlinear programming techniques, dynamic programming methods, variational methods, stochastic approximation procedures, optimization in simulation and optimization in function spaces.

Classical optimization techniques involve preliminaries, necessary and sufficient conditions for an extremum and constrained optimization (Lagrange Multipliers).

Details of the classical optimization techniques are presented as follows:

Optimization and inequalities: Classical inequalities and matrix inequalities.

Numerical methods of optimization**:** Numerical evaluation of roots of equations, direct search methods, gradient methods, convergence of numerical procedures, nonlinear regression and other statistical algorithms.

Linear programming techniques: Linear programming problem, standard form of the linear programming problem, simplex method, Karmarkar's algorithm, zero-sum two Person finite-games and linear programming, integer programming and statistical.

Nonlinear programming methods: Statistical examples, Kuhn-Tucker conditions, quadratic programming, convex programming, applications, statistical control of optimization, stochastic programming and geometric programming.

Dynamic Programming Methods**:** Regulation and control, functional equation and principles of optimality, dynamic programming and approximation, patient care through dynamic programming and Pontryagin maximum principle.

Variational Methods: Euler-Lagrange equations, Neyman-Pearson technique, robust statistics and variational methods and penalized maximum likelihood estimates.

Stochastic approximation procedures**:** Robbins-Monro procedure, general case, Kiefer-Wolfowitz procedure and stochastic approximation and filtering. (Wikipedia, 2018).

METHODOLOGY

Optimization(optimality) will be demonstrated by examples in systems, designs

and decision situations using the various classical optimization techniques.

THE INDUSTRIAL SYSTEM EXAMPLES.

1. A calculator company produces scientific and graphic calculators. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphic calculators each day. Due to the limitations in production capacity, no more than 200 scientific and 170 graphic calculators can be made daily. A total of at least 200 calculators must be shipped each day to satisfy a shipping contract. Each scientific calculator sold results in a loss of #3, but each graphic calculator produces a profit of #6. Formulate the mathematical model of the optimization problem. What are the entities and activities of the transaction?

2. A toy company manufactures two types of dolls, a basic version, A and a deluxe version, B. Each doll type B takes twice as long to produce as one of type A. The company will have time to make a maximum of 20 per day if it produces only the basic version. The supply of plastic is sufficient to produce 15 dolls per day of both versions combined. The deluxe version requires a fancy dress of which there are 6 per day available. The company makes a profit of 3k and 5k per doll respectively on dolls A and B. Make a mathematical model presentation of the transaction. How many of each doll should be produced per day to maximize profit?

SOLUTIONS

1.Let X_1 = No. of scientific calculators produced
X_2 = No. of graphic calculators produced
R = Revenue relation (Net profit)
The optimization model is given by
Maximize $R = -3X_1 + 6X_2$ (Objective function)

Subject to $100 \leq X_1 \leq 200$
$80 \leq X_2 \leq 170$
$X_1 + X_2 \geq 200$

The entities are the company, calculators (scientific and graphic) and customers while the activities are production and shipping of calculators.

2. The mathematical model of the optimization problem is given by
Maximize $f(X) = 3X_1 + 5X_2$ (Objective)
Subject to $X_1 = 2X_2$
$X_1 \leq 20$
$X_1 + X_2 \leq 15$
$X_2 \leq 6$
X_1 = No. of doll A and X_2 = No. of doll B. $X_1, X_2 \geq 0$.
ii. Solving graphically or simultaneously we obtain
$X_1 = 10$, $X_2 = 5$ and $f(X) = 3(10) + 5(5) = 55$

REMARK

The first example deals with formulation of the model indicating the objective function and constraints (entities and activities) of an optimization problem in a typical production system. The second example goes further to give optimal solution to an optimization problem in a typical manufacturing system.

QUEUING MODELS INVOLVING PROBABILITY (Humphreys, 1991)

The arrival rate and servicing time are not constant in practice. If the number of arrivals per period follows a Poisson distribution with λ average arrivals per period and the number of items serviced per period also follows a Poisson distribution with μ > λ then minimum cost per period is obtained at

$$C_{t.opt.} = 3C_w$$

$$\lambda'_{opt.} = C_w/C_f$$

$$\mu'_{opt.} = 2\frac{C_w}{C_f}$$

Where $\lambda'_{opt.}$ and $\mu'_{opt.}$ are specific values of independent variables λ' and μ' at which the lowest cost, $C_{t.opt.}$ of the system occurs. C_w and C_f are an item cost of waiting and cost of servicing respectively.

Example

The cost of waiting per period is #4 and the cost per hour for servicing an item in a service centre which can handle one item in 1 hour is #2. If the arrival rates follow Poisson distribution, find the lowest cost policy.

Solution

$C_{t.opt.} = 3C_w = 3(4) = \#12$ per period

$\lambda'_{opt.} = C_w/C_f = 4/2 = 2$ items per period

$$\mu'_{opt.} = 2\frac{C_w}{C_f} = 2\frac{4}{2} = 4 \text{ items per period}$$

REMARK
The example shows optimality in queuing system involving probability.

USING DUALITY PRINCIPLE TO SOLVE OPTIMIZATION PROBLEMS(Ekeocha,2018)

The duality principle provides that optimization problems may be viewed from either of two perspectives, the primal problem or the dual problem. The solution to the dual problem provides a lower bound to the solution of the primal (minimization) problem. However, the optimal values of the primal and dual problems need not be equal. Their difference is called the duality gap. For convex optimization problems, the duality gap is zero under a constraint qualification condition. In other words, given any linear program, there is another related linear program called the dual. In this paper, an understanding of the dual linear program will be developed. This understanding will give important insights into the algorithm and solution of optimization problem in linear programming. Thus, the main concepts of duality will be explored by the solution of simple optimization problem.

WORKED EXAMPLES
Two examples will be presented to illustrate the relationship between primal and dual linear programs, showing that linear programming problems can be solved from two different perspectives.

Example 1.

A university staff is contemplating what to purchase for his family for lunch in the school cafeteria's bakery. It appears he came a bit late and there are just 2 choices of food left, wheat bread which costs N12 each, and chocolate bread which costs ₦20 each. The bakery is service-oriented and is happy to let the staff purchase a fraction of an item if he wishes. The bakery requires 7 ounces of chocolate to make each wheat bread and 12 ounces of chocolate for the chocolate bread. 6 ounces of sugar are needed for each wheat bread and 8 ounces of sugar for each chocolate bread. The staff has decided that he needs at least 100 ounces of sugar and 120 ounces of chocolate. He wishes to optimize his purchase by finding the least expensive combination of wheat and chocolate bread that meet these requirements.

Solution
Primal:

Let x_1 and x_2 be the wheat bread and chocolate bread respectively. Then the optimization problem can be formulated as follows:

Minimize $Z = 12x_1 + 20x_2$

Subject to $6x_1 + 8x_2 \geq 100$

$7x_1 + 12x_2 \geq 120$

$x_1, x_2 \geq 0$

Expressing in standard form we have

$Z = 12x_1 + 20x_2 + 0s_1 + 0s_2$

Subject to $6x_1 + 8x_2 - s_1 = 100$

$7x_1 + 12x_2 - s_2 = 120$

$x_1, x_2, s_1, s_2 \geq 0$

Table 1: 1ST ITERATION

		C_j	12	20	0	0	M	M		
F.R	C_b	Basis	X_1	X_2	S_1	S_2	A_1	A_2	B	θ
2/3	M	A_1	6	8	-1	0	1	0	100	25/2
	M	A_2	7	(12)	0	-1	0	1	120	10←
		Zj	13M	20M	-M	-M	M	M	220M	
		$C_j - Z_j$	12–13M	20-20M	M	M	0	0		
				↑						

Table 2: 2ND AND 3RD ITERATION

		C_j	12	20	0	0	M		
F.R	C_b	Basis	X_1	X_2	S_1	S_2	A_1	B	θ
	M	A_1	(4/3)	0	-1	2/3	1	20	15←
7/16	20	X_1	7/12	1	0	-1/12	0	10	120/7
		Z_j	35/3+4/3M	20	-M	-5/3+2/3M	M	200+20M	
		$C_j - Z_j$	1/3-4/3M		M	5/3-2/3M	0		
			↑						

		Cj	12	20	0	0			
F.R	C_b	Basis	X_1	X_2	S_1	S_2		B	θ
	12	X_1	1	0	¾	1/2		15	
	20	X_2	0	1	7/16	-3/4		5/4	
		Z_j	12	20	-1/4	-9		205	
		$C_j – Z_j$	0	0	¼	9			

Optimal solution is $x_1 = 15$, $x_2 = 5/4$; Zmin = ₦205. Hence 15 loaves of wheat bread and 1¼ loaf of chocolate bread will optimize his cost and also give him the required amount of sugar and chocolate he desires.

Dual:
We now adopt the perspective of the wholesaler who supplies the baker with the chocolate and sugar needed to bake the bread. The baker tells the supplier all he needed and also showed him the list drafted from the university staff's demand. The supplier now solves the following optimization problem. How can I set the prices per ounce of chocolate and sugar so that the baker will buy from me so that I will maximize revenue? The baker will buy only if the total cost of raw materials for wheat bread is below N12; otherwise he runs the risk of making a loss if the staff opts to buy wheat bread and also for the chocolate bread. This restriction imposes the following constraints on the price.

Here we will be using the duality technique to form a new model from the primal problem model.

Then we have:

Maximize: $Y = 100u_1 + 120u_2$

Subject to $6u_1 + 7u_2 \leq 12$

$8u_1 + 12u_2 \leq 20$

Where u_1 and u_2 is price for sugar and chocolate respectively
$u_1, u_2 \geq 0$

Now using the simplex table to optimize we have

Table 3: 1ST ITERATION

		Cj	100	120	0	0		
F.R	C_b	Basis	u1	u2	s1	s2	B	θ
7/12	0	S1	6	7	1	0	12	12/7
	0	S2	8	(12)	0	1	20	20/12→
		Yj	0	0	0	0	0	
		Cj – Yj	100	120 ↑	0	0		

Table 4: 2ND ITERATION

		Cj	100	120	0	0		
F.R	C_b	Basis	U_1	U_2	S_1	S_2	B	θ
	0	S_1	(4/3)	0	1	-7/12	1/3	¼
1/2	120	U_2	2/3	1	0	1/12	5/3	5/2→
		Y_j	80	120	0	10	200	
		$C_j - Y_j$	20 ↑	0	0	-10		

Table 5: 3RD ITERATION

	C_j	100	120	0	0	
C_b	Basis	U_1	U_2	X_1	X_2	B
100	U_1	1	0	¾	-7/16	¼
120	U_2	0	1	-1/2	9/24	9/6
	Y_j	100	120	15	5/4	205
	C_j-Y_j	0	0	-15	-5/4	

The wholesaler's optimal price for an ounce of sugar and an ounce of chocolate is N0.25 and N1.5 respectively. Looking closely, we discover that the result is the same optimal amount of N205. This clearly indicates that the dual problem could also be a solution to the primal problem.

Example 2

The simplex method is employed as the solution method as follows:

- Write the problem in matrix form starting with the first constraint as the first row and the objective function as the last row.
- Find the transpose of the primal matrix to get the dual matrix.
- The first row of the transpose translates to the dual constraints and the last row becomes the new objective function.
- Observe that the inequality signs of the constraints change after dualization and the objective function changes from maximize to minimize or vice versa depending on the objective function.

Minimize $G = 11x + 7y$

Subject to $x + 2y \geq 10$

$3x + y \geq 15$

Solution

Primal

Table 6: Primal Problem

1	2	10
3	1	15
11	7	0

Table 7: 1ST Iteration

Basis	x	y	S_1	S_2	A_1	A_2	b	Ratio
A_1	1	2	-1	0	1	0	10	10/1
A_2	3		1	0	-1	1	15	15/3
Z	11-4M	7-3M	M	M	0	0	0	

Table 8: 2ND Iteration

Basis	x	Y	S_1	S_2	A_1	A_2	b	Ratio
A_1	0	5/3	-1	0	1	0	10	10/1
x	1	1/3	0	-1/3	0	-	5	0
Z	0	$\frac{10-5M}{3}$	M	$\frac{11}{3}$ - M	0			

Table 9: 3RD iteration

Basis	x	y	S_1	S_2	A_1	b	Ratio
y	0	1	$-\frac{3}{5}$	$\frac{1}{5}$		3	
x	1	0	$\frac{1}{5}$	$-\frac{2}{5}$		4	
Z	0	0	2	3		65	

X = 4, y =3 and G = 65.

Dual

Maximize   Z = 10u + 15v

Subject to $u + 3v \leq 11$

$2u + v \leq 7$

Table 10: The Dual Problem

1	3	11
2	1	7
10	15	0

Table 11: 1ST Iteration

	u	V	S_1	S_2	b	Ratio
S_1	1	3	1	0	11	11/3
S_2	2	1	0	1	7	7/1
Z	-10	-15	0	0	0	

Table 12: 2ND Iteration

	u	V	S_1	S_2	b	Ratio
v	1/3	1	1/3	0	11/3	11
S_2	5/3	0	1/3	1	10/3	2
Z	-5	0	5	0	55	

Table 13: 3RD Iteration

	u	V	S_1	S_2	b	Ratio
v	0	1	6/15	-1/5	3	
u	1	0	-1/5	3/5	2	
Z	0	0	6	5	65	

u = 2, v = 3 and Z = 65.

REMARK

The examples show that the optimization problem can be viewed from two perspectives namely the primal problem and the dual problem. The simplex method (iteration) is used to achieve optimality in the examples.

The examples also show that the duality gap is zero. This is an indication of convex optimization problem. In other words, the variables and the constraints of the optimization problem for the examples are small and equal in number.

Sometimes it is easier to solve the dual especially when it is used to detect primal infeasibility which is the consequence of weak duality. This is the case when the primal problem has many constraints and few variables. It can be converted into a dual problem with few constraints and many variables. Fewer constraints have been shown to be preferred in linear programs because the basis matrix is an n by n matrix, where n is the number of constraints. Thus, the fewer the constraints, the smaller the size of the basis matrix, and thus the fewer computations required in each iteration of the simplex method.

CONCLUSION

In the industrial system, the examples show the formulation of the objective function and constraints in a typical production outfit as well as deriving optimality in a problem of a typical manufacturing system employing graphical or simultaneous equation method.

In the queuing system, stochastic approximation is used to obtain optimality of queuing model involving probability.

In duality, two approaches to optimization problem namely the primal problem and the dual problem are considered. The latter is used to check the accuracy of the former especially in the case of zero duality gap.

Duality in linear programming has far reaching consequence of economic nature which assists managers answer questions about alternative courses of action and their relative values. It therefore provides efficient algebraic technique that enhances the study of the dynamic behavior of optimization problems.

It is observed that the systems considered in the paper required specific optimization techniques to achieve optimality.

REFERENCES

1. Britannica.com: https://www.britannica.com/topic/optimization
2. Ekeocha, RJO (2018) The Use of Duality Principle to Solve Optimization Problems, International Journal of Recent Contributions from Engineering Science & IT (iJES), vol. 6, no.1, 2018, pp. 33-42.
3. Humphreys, K.K. (1991) Jelen's Cost & Optimization Engineering, 3RD Edition, Publisher: McGraw Hill Inc.
4. Merriam Webster: https://www.merriam-webster.com/dictionary/optimization
5. Wikipedia, 2018: https://en.wikipedia.org/wiki/Optimization_problem