Optimality in Inventory and Machinery Replacement Models

Rowland Jerry Okechukwu Ekeocha[1]✎

[1]Covenant University, Ota Nigeria

Abstract: In my first article, optimization of industrial system was achieved by the use of the simultaneous equation method while stochastic approximation was employed in obtaining optimality of the queuing system. In this paper, differential calculus and repeated trials were employed to located the optimum lot size of the inventory model while the optimization, dynamic programming and the economic lot size methods were required to obtain optimality of the replacement/rehabilitation model. If the first article created an impression that optimality is limited to some systems like the industrial and queuing systems the extension of optimality to the inventory and replacement/rehabilitation model shows that optimality is achievable in every system/model provided appropriate optimization technique is employed.

Keywords: Optimality, Optimization Technique, Models

Introduction
Optimization is defined as the mathematical procedures involved in effecting optimality. It is also a collection of mathematical principles and methods used for solving quantitative problems in many disciplines. (Merriam Webster). The model of a system is a replica (physical or mathematical) which has all the properties (attributes) and function of the system. According to Singh (2009), formal models can be classified as physical, mathematical and computer models. The physical model is the scaled down model of the actual system which has all its properties. Mathematical models represent a system with mathematical equations while computer models utilize computers to numerically evaluate mathematical models. System and model may be interchanged in this write-up. My first article (Ekeocha, 2019) on optimality focused on optimization of industrial and queuing systems. The extension of optimality to inventory and machinery replacement models indicates that optimality is achievable in virtually every situation provided appropriate optimization technique is employed.

Methodology
Optimality in inventory and machinery replacement models will be addressed with worked examples employing appropriate optimization technique.

Inventory Model (Humphreys, 1991)
This is a stock of physical assets having value which can be material, money or labour. Material inventory can be raw material, tools and accessories including spare parts used in production, unfinished or in-process inventory and finished products. Inventories are maintained at a cost to gain advantages having monetary value, such as avoiding a shutdown due to a temporary absence of supplies or permitting uniform production for a reasonable supply or demand.

Inventory models which are usually mathematical (sometimes symbols) may be deterministic, for which all information and values are treated as being definite or probabilistic (stochastic), for which uncertainty in some values is recognized and considered.

In the general inventory system, ordinates represent the number of units available in stock against time in the abscissa. Consider quantity Q units received at the beginning of the period T_1. The use during this period U_1 is subtracted from the stock to give a lower stock for the second period, etc. At the end of the fourth period T_4 the level of inventory reaches the reorder level R. At that time an order equal to Q unit is placed, which is eventually received at the beginning of the period. The procurement time or lead time T_i is the time required in period to receive an order after it is placed. During the procurement time the level of stock may go to zero, creating a shortage for which a penalty can arise. The objective is to minimize the total cost of the system, which is the minimum cost per period (not per cycle). Some costs are connected with the cycle, such as placing and receiving an order, whereas other costs are connected with a period, such as storage costs. Confusion between a period and a cycle is the most common source of error for a beginner.

✎   **Rowland Jerry Okechukwu Ekeocha (Correspondence)**
☎   +

Economic Lot Size

The following assumptions are made for the simplest case:

- All items are delivered simultaneously at the time the stock becomes zero
- The cost of placing and receiving an order is constant and is independent of the lot size
- Usage is at a constant rate

- No safety stock is provided
- No shortage is allowed
- The warehousing cost is calculated on a continuous basis and at any instant of time is proportional to the number of items in the inventory.
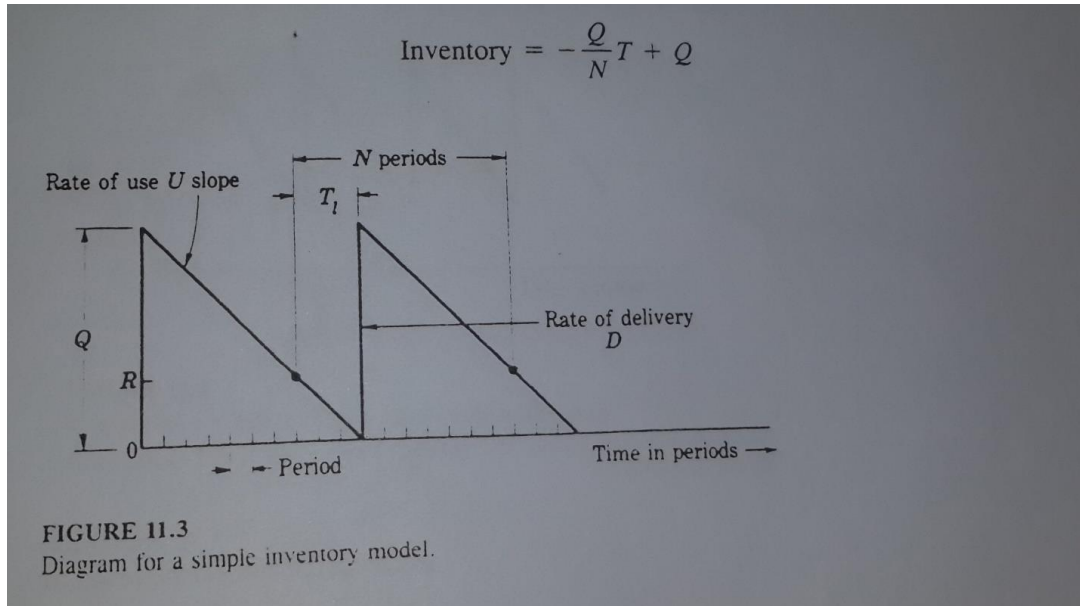

$$\text{Inventory} = -\frac{Q}{N}T + Q$$

FIGURE 11.3
Diagram for a simple inventory model.

Fig. 11.3 illustrates the schematic diagram for economic lot size. Let C_0 be the cost of placing and receiving an order, C_h the storage or holding cost per item per period, $M = Q$ the maximum inventory where Q is the lot order, and U the rate of use in items per period. Consider a time element T unit of time from zero-time lasting dT units of time. The inventory at any time is a straight-line function of time and is Q at T = 0 and 0 at T = N where N is the duration of the cycle. Then

$$\text{Inventory} = -\frac{Q}{N}T + Q$$

The number of periods of time N that the lot Q will last is given by

$$N = \frac{items}{items/\,period} = \frac{Q}{U}$$

The inventory at any time is given by
Inventory = -UT + Q
The total cycle cost is given by

$$C_{cycle} = C_0 + \int_{0}^{N=Q/U} C_h(-UT + Q)\,dT = C_0 + C_h\left(-\frac{UQ^2}{2U^2} + Q\frac{Q}{U}\right)$$

$$= C_0 + \frac{C_h Q^2}{2U}$$

The function to be optimized is the cost per period C_t and with

$$\text{Cycle/period} = \frac{item/\,period}{item/\,cycle} = \frac{U}{Q}$$

The cost per period becomes

$$C_t = \frac{Naira}{cycle}\frac{cycle}{period} = (C_0 + \frac{C_h Q^2}{2U})\frac{U}{Q} = C_0\frac{U}{Q} + \frac{C_h Q}{2}$$

Differentiating and setting to zero, we obtain

$$\frac{dC_t}{dQ} = -C_0\frac{U}{Q^2} + \frac{C_h}{2} = 0$$

Finally,

$$Q_{opt.} = \sqrt{\frac{2C_0 U}{C_h}}$$

The lead time T_i multiplied by the rate of use per period U gives the reorder level, R. Thus
$R = T_i\,U$
By placing $Q_{opt.}$ in C_t, the optimum period cost is obtained,

$$C_t = C_0 \frac{U}{\sqrt{2C_0 U / C_h}} + C_h \sqrt{\frac{C_0 U}{C_h}} = \sqrt{2C_h C_0 U}$$

Example:
The demand for an item 100/day. The cost of placing and receiving an order is #50 with an infinite delivery rate. The storage cost is #10 per item per year based on the actual inventory at any time. No shortage is allowed. The reorder level is an 8-day delivery. Find the optimum stock, the optimum period cost and the reorder level.

Solution:
   i. The storage charge per item per day is 10/365 = 0.0274. Thus, the optimum stock is

$$Q_{opt.} = \sqrt{\frac{2(50)(100)}{0.0274}} = 604$$

   ii. The optimum period cost is given by
$$C_t = \sqrt{2(0.0274)(50)(100)} = 16.55 / day$$

   iii. The reorder level is given by
$$R = 8(100) = 800$$

Safety Stock:
This is added to the inventory to ensure that a shortage in an item does not occur. The inventory model becomes
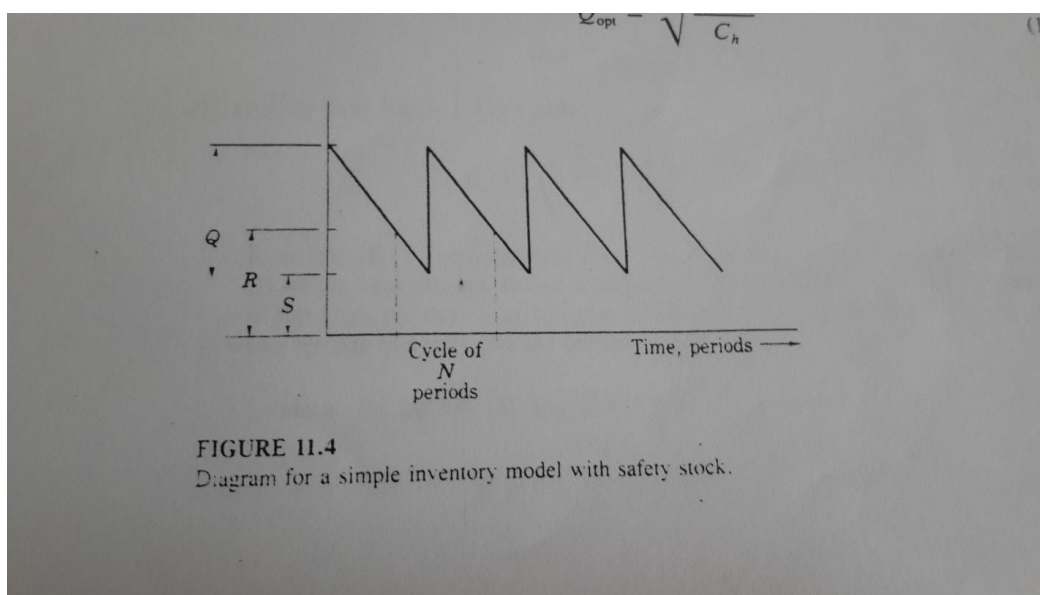
(fig.11.4)

FIGURE 11.4
Diagram for a simple inventory model with safety stock.

Inventory = -UT + Q + S (S = Safety Stock)
The cost per cycle (following individual steps established previously)

$$C_{cycle} = C_0 + C_h(\frac{Q^2}{2U} + S\frac{Q}{U})$$

The cost per period becomes

$$C_t = C_0 \frac{U}{Q} + C_h \frac{Q}{2} + SC_h$$

The optimum order for minimum cost becomes
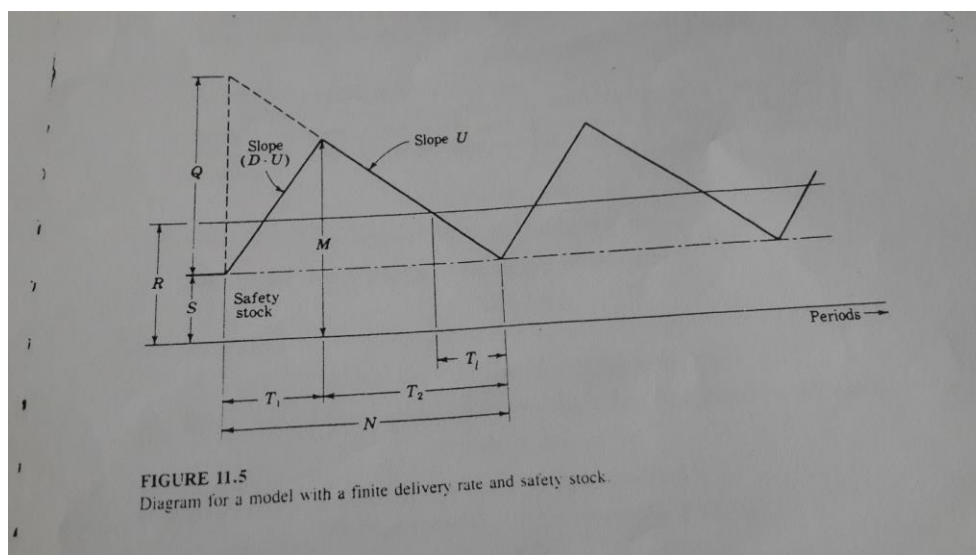
$$Q_{opt.} = \sqrt{\frac{2C_0 U}{C_h}}$$

The optimum period cost becomes

$$C_t = \frac{\sqrt{2C_h C_0 U} + SC_h}{\sqrt{2C_h C_0 U}}, when, S = 0$$

Finite Rate Delivery
Finite rate delivery model is associated with manufacturing.

FIGURE 11.5
Diagram for a model with a finite delivery rate and safety stock.

The schematic diagram (fig.11.5) illustrates the model concept. C_0 is a setup cost and D item per unit of time is a manufacturing rate. The use rate in items per unit of time is U. The value for Q is the total number of items produced and used in the cycle of $(T_1 + T_2)$ periods. The safety stock is S items and M is the maximum inventory. In time T_1 the inventory builds up by an amount (M − S) at a rate (D − U). Thus

M − S = T_1 (D − U) and Q = T_1D (Since Q is made at D rate in timeT_1)

Eliminating T_1 gives

$$M - S = Q\,\frac{D-U}{D}$$

But Q = UN (Q is used in N period at U rate), then

$$M - S = UN\frac{D-U}{D}$$

In this model (M − S) is linear with time on both sides of the maximum inventory. The average inventory for the total N periods is half (M − S) plus the safety stock.

$$\text{Average inventory} = \frac{UN}{2}\frac{D-U}{D} + S$$

The average inventory is held at a cost C_h per item per period. The cost per period is

$$C_t = [C_0 + (\frac{UN}{2}\frac{D-U}{D} + S)\,C_hN]\,\frac{naira}{cycle}\frac{1}{N}\frac{cycle}{period}$$

$$= \frac{C_0}{N} + \frac{UN}{2}\frac{D-U}{D}C_h + C_hS$$

Differentiating, we obtain,

$$\frac{dC_t}{dN} = \frac{C_0}{N^2} + \frac{C_hU}{2}\frac{D-U}{D} = 0$$

$$N = \sqrt{\frac{2C_0D}{C_hU(D-U)}}$$

Thus, the optimum stock is

$$Q_{opt.} = \sqrt{\frac{2C_0UD}{C_h(D-U)}} \qquad \text{(Since Q = UN)}$$

And the minimum period cost is

$$C_t = \sqrt{\frac{2C_hC_0U(D-U)}{D}} + C_hS$$

Example:
The setup cost for a small operation is #600 and the storage cost is #0.40 per item per day. Usage is uniform at 300 items per day and the production rate is 500 items per day. A safety stock of 100 units is required. Find the optimum production lot size that gives minimum cost.

Solution:
i. The optimum lot size is given by

$$Q_{opt.} = \sqrt{\frac{2(600)(300)(500)}{(0.40)(500-300)}} = 1500$$

ii. The minimum period cost is given by

$$C_t = \sqrt{\frac{2(0.40)(600)(300)(500-300)}{500}} + 0.40(100) = 280/day$$

iii. The production time is given by

$$T_i = \frac{Q}{D} = \frac{1500}{500} = 3 days$$

iv. The cycle time is given by

$$N = \frac{Q}{U} = \frac{1500}{300} = 5 days$$

Example: (Setting Up Inventory Problem)
The demand of an item is uniform at 1000 items per day. Delivery when placing an order is immediate. The cost for placing and receiving an order size is given by

$$C_0 = 13000 + 5Q^{0.5}$$

It costs #0.002/day for each item in storage. In addition, there is an antispoilage cost while in storage which is proportional to the age of the inventory multiplied by the number of units in the inventory at the time. If the inventory is 20 days old and there are 8000 items in stock, the antispoilage cost would be

(0.000015) (20) (8000) #/day, where 0.000015 is a constant.
Find the optimum lot size.
Solution:
Let a cycle last N day. The lot size will then be Q = 1000N and

$$C_0 = 13000 + 5(1000N)^{0.5} = 13000 + 158.11N^{0.5}$$

The inventory declines linearly from 1000N to zero in N days. The inventory at any time is
Inventory = 1000(N – T), where T is in days.

The cost for antispoilage at that time is given by
(0.000015) T [1000(N – T)] = 0.015(NT – T²) #/day

The cost per cycle $= 13000 + 158.11N^{0.5} + \int_0^N 0.002[1000(N - T)]dT + \int_0^N 0.015(NT - T^2)dT$

Integrating, we obtain,
$13000 + 158.11N^{0.5} + N^2 + 0.0025N^3 = 13000N^{-1} + 158.11N^{-0.5} + N + 0.0025N^2 = 0$

By repeated trails, the minimum occurs at N = 96.4 for which the optimum lot size
$Q_{opt.} = 1000N = 96400$ items.

Remark
The inventory model is a differentiable function and thus requires differential calculus in locating the optimum solution. The third example (Setting Up Inventory Problem) employs repeated trials to locate optimum lot size. The examples indicate the importance of appropriate technique in the solution of specific optimization problem.

Machinery Replacement/Refurbishment Model
A machinery replacement/refurbishment model (Ekeocha, 2011) is given by

$$K(T) = \frac{1}{R}[Q + B(t)R - S(t)R^t]$$

And

$$K(T) = \frac{1}{R}[Q + (b_1 + b_2 t)R - Q(1 - d)^t R^t]$$

Where
K(T)   = Present value of total cost
B(t)   = Time-dependent (increasing) Maintenance cost
S(t)   = Salvage value (Deterioration and time dependent)
  Q   = Cost of machine (constant)
  R   = Discount factor $\left(\frac{100}{r+100}\right)$
  r   = Rate of return on replacement investment
  d   = Deterioration rate

As usual, the objective is to find T that will minimize K (T). The time, T that gives minimum present value of total cost, K (T) corresponds to the replacement date of the machinery. Conversely, the time, T that produces maximum K(T) corresponds to the repair/refurbishment of the machinery.

Example
Consider a machine with the following technical details (ASA, 2000)
Name of machine: Front-end Loader
Model: Y
Year of manufacture: 1995
Cost: $60,000.00
Resale Value (Active Market): $40,000.00 (5th year)
Deterioration rate: 25%
Capitalization rate: 40%
Minor maintenance: $2,500.00 annually
Major maintenance: (i) $10,000.00 (2nd year)
                        (ii) $3,500.00 (3rd year)
                        (iii) $6,000.00 (4th year)
                        (iv) $20,000.00 (5th year)

Solution: The tabular presentation is shown in Table 1

Table 1: Data for model Y, Front-End Loader

T	Q	b_1	b_2t	S(t)
1	60,000	2500	-	-
2	60,000	2500	10,000	-
3	60,000	2500	3,500	-
4	60,000	2500	6,000	-
5	60,000	2500	20,000	40,000
6	60,000	2500	-	-
7	60,000	2500	-	-

Table 2: Enumeration of Data for Hypothetical front-end Loader

t	Q	R	R^t	b_1	b_2t	B(t)	B(t)R	d	S(t)	$S(t)R^t$	K(T)
1	60,000	0.71	0.71	2500	-	2500	1785	0.33	-	-	86533
2	60,000	0.71	0.51	2500	10,000	12500	8925	0.50	-	-	96533
3	60,000	0.71	0.36	2500	3500	6000	4284	0.50	-	-	90033
4	60,000	0.71	0.26	2500	6000	8500	6069	0.83	-	-	92533
5	60,000	0.71	0.19	2500	20,000	22500	16,065	0.83	40,000	7424	96136
6	60,000	0.71	0.13	2500	-	2500	1785	0.83	40,000	5300	79111*
7	60,000	0.71	0.09	2500	-	2500	1785	1.00	40,000	3784	81234
10	60,000	0.71	0.03	2500	-	2500	1785	1.00	40,000	1376	84606

Table 3: Predicted K (T) for Front-End Loader

T	Q	R	R^t	B(t)	B(t)R	d	S(t)	$S(t)R^t$	K(T)
1	60000	0.71	0.71	6653.82	4750.83	0.33			89437
2	60000	0.71	0.51	16653.64	1189.70	0.50			99298
3	60000	0.71	0.36	10153.47	7249.58	0.50			92888
4	60000	0.71	0.26	12653.29	9034.45	0.83			95353
5	60000	0.71	0.19	26653.11	19030.32	0.83	8.52	1.58	109158
6	60000	0.71	0.13	6652.93	4750.19	0.83	1.45	0.19	89435*
7	60000	0.71	0.09	6652.75	4750.07	1.00	0	0	89435
10	60000	0.71	0.03	6652.22	4749.69	1.00	0	0	89435

MAINTEANCE COST REGRESSION LINE,

B(t) = 6654 – 0.1778t

b_1 = 6654

b_2 = -0.1778

b_2t = Cost of Refurbishment in any year of refurbishment (Overhaul)

TABLE 4.:   K(T)   against Time (Measured L and Predicted M)

T	1	2	3	4	5	6	7	8
K(T)L	86533	96533	90033	92533	96136	79111	81234	84606
K(T)M	89437	99298	92888	95353	109158	89435	89435	89435
%Dev.	-3.24	-2.78	-3.07	-2.96	-11.93	-11.54	-9.17	-5.40

The percentage deviation (%Dev.) between the measured and predicted K(T) is less than 12% which is an indication of a low error margin.
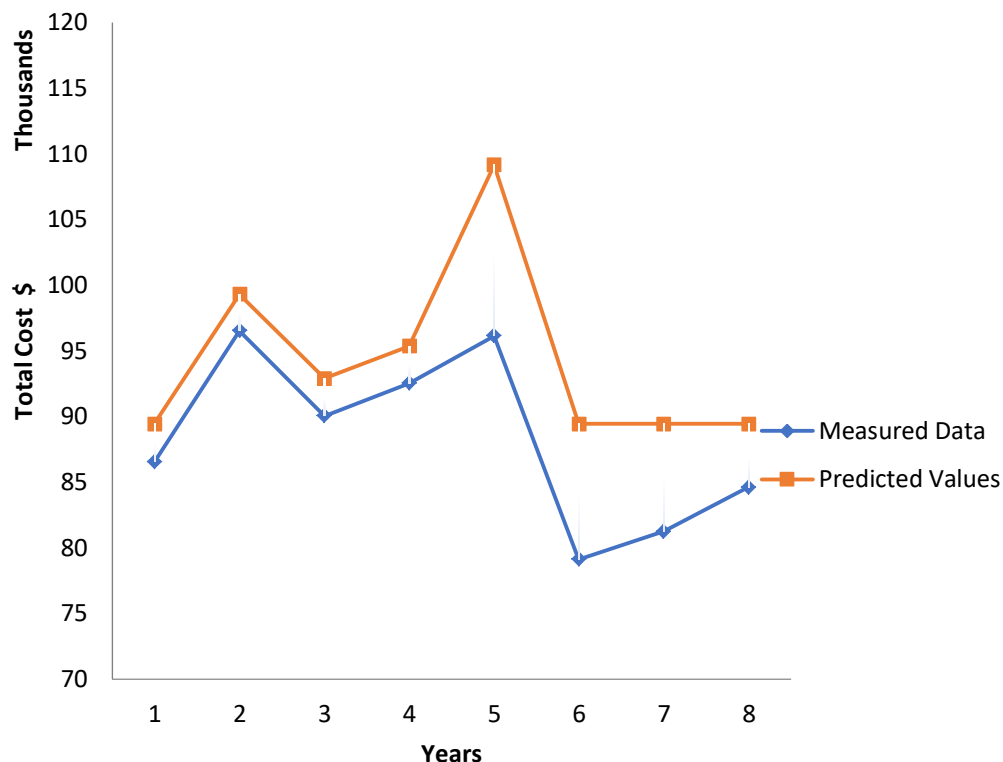
Fig.1: Comparison of the measured and predicted total cost K(T) of the Loader

The result from the tables and graph shows a replacement date of 6th year. (Repair/Refurbishment in the 2nd and 5th years).

Remark

The model employs simple and straight forward solution techniques like optimization, dynamic programming and economic lot size inventory control method. The model exhibits the characteristics of overlapping subproblems and optimal substructure hence the use of these techniques for optimality. The model is calibrated with a set of data for the machine from literature while verification and comparison are performed with another set of field data from companies. The graphical presentation (fig. 1) confirms the suitability of the solution techniques.

Conclusion

Differential calculus is a suitable solution technique for the inventory model because it is a differentiable function. In some cases, repeated trials are employed in locating the optimum lot size of the inventory model. In the case of the machinery replacement/rehabilitation model, the optimization, dynamic programming and economic lot size method is suitable for optimality because the model exhibits the characteristics of overlapping subproblems and optimal substructure. In fact, the salvage value subproblem is a differentiable function like the inventory model and the differential calculus can be employed to obtain the initial replacement/rehabilitation date. Graphical presentation can also be used to complement the optimization techniques employed in the models.

Generally, it can be said that optimality is achievable for every system/model provided appropriate optimization technique is employed.

References

1. American Society of Appraisers (ASA, 2000), Valuing machinery and technical assets, Washington DC.
2. Ekeocha RJO, Odukwe AO, Agunwamba JC (2011) Machinery Replacement Problems Model, World Journal of Engineering and Pure and Applied Science, vol.1, no.3, pp. 81 – 87.
3. Ekeocha, RJO. (2019) Optimization of Systems, International Journal of Sciences, 03 (2019) 117-124. DOI:10 18483/ijSci. 1964.
4. Humphreys, K.K. (1991) Jelen's Cost & Optimization Engineering, 3RD Edition, Chapter 11, Publisher: McGraw Hill Inc. New York.
5. Merriam Webster: https://www.merriam-webster.com/dictionary/optimization.
6. Singh VP, (2009) System Modeling and Simulation, New Age International Publishers.